Graph Theory

Homework assignment #4

Due date: Sunday, January 14, 2018

Problem 1. Let G be a graph obtained from $K_{n,n}$ by replacing one of its edges by a path of length two. Show that $\chi'(G) = \Delta(G)+1$, but if e is any edge of G, then $\chi'(G-e) = \Delta(G-e)$.

Problem 2. Suppose that a cubic (3-regular) graph G has exactly one edge-colouring with $\chi'(G)$ colours, up to a permutation of the colours. Show that $\chi'(G) = 3$ and that G has exactly three Hamilton cycles.

Problem 3. Show that every red/blue-colouring of the edges of K_{6n} contains n vertex-disjoint triangles with all $3n$ edges of the same colour.

Problem 4. Given two graphs G and H, let $R(G,H)$ denote the smallest integer n such that every red/blue-colouring of the edges of K_n contains either a blue copy of G or a red copy of H. Determine $R(K_s, P_t)$ for all positive s and t, where P_t is the path with t vertices.

Problem 5. Prove that $R(n\cdot K_2, n\cdot K_2) = 3n-1$ for every n, where $n\cdot K_2$ denotes n independent edges.

Problem 6. Prove that for every tree T and integers $k \geqslant 2$ and $g \geqslant 3$, there exists a graph G without cycles of length up to g and such that every k-colouring of the edges of G contains a monochromatic copy of T.

Problem 7. Let H be the graph with four vertices and five edges. Prove that $\mathrm{ex}(n,H) = \mathrm{ex}(n,K_3)$ for every $n \geqslant 4$.

Problem 8. Prove that if $n \geqslant 5$, then every graph of order n with $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles with exactly one common vertex.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Let G be an n-vertex graph with $\lfloor n^2/4 \rfloor - t$ edges and no triangle. Show that one can make G bipartite by deleting from it at most t edges.

Exercise 2. Suppose that n, m, and t are the numbers of vertices, edges, and triangles (respectively) of a graph. Show that $3t \geqslant 4m^2/n - mn$. Note that this implies that $\mathrm{ex}(n,K_3) \leqslant n^2/4$.