

# Graph Theory

## Homework assignment #3

Due date: Sunday, December 27, 2020

**Problem 1.** Let  $G$  be a bipartite graph with bipartition  $V(G) = A \cup B$ . Define

$$\delta(A) = \max_{S \subseteq A} (|S| - |N(S)|).$$

Prove that the maximum size of a matching in  $G$  is  $|A| - \delta(A)$ .

**Problem 2.** An  $n \times n$  Latin square (resp.  $r \times n$  Latin rectangle) is an  $n \times n$  (resp.  $r \times n$ ) matrix with entries in  $\{1, \dots, n\}$  such that no two entries in the same row or column are the same. Prove that if  $r < n$ , then every  $r \times n$  Latin rectangle may be extended to an  $n \times n$  Latin square.

**Problem 3.** Let  $G$  be a connected graph with an even number of edges. Use Tutte's theorem to prove that the set of edges of  $G$  can be partitioned into pairwise disjoint paths of length 2.

**Problem 4.** Let  $v$  be a vertex of a connected graph  $G$  and, for  $r \geq 0$ , let  $G_r$  be the subgraph of  $G$  induced by the vertices at distance exactly  $r$  from  $v$ . Show that

$$\chi(G) \leq \max \{ \chi(G_r) + \chi(G_{r+1}) : r \geq 0 \}.$$

**Problem 5.** Let  $G$  be a graph on  $n$  vertices. Prove that  $\chi(G) \cdot \chi(\bar{G}) \geq n$ .

**Problem 6.** Suppose that a graph  $G$  is a union of  $k$  trees. Prove that  $\chi(G) \leq 2k$ .

**Problem 7.** Suppose that every pair of odd cycles in a graph  $G$  has a common vertex. Show that  $\chi(G) \leq 5$ .

**Problem 8.** Prove that every graph  $G$  with  $\chi(G) \geq 3$  contains a cycle of length at least  $\chi(G)$ .

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**Please do NOT submit written solutions to the following exercises:**

**Exercise 1.** Prove that every bipartite graph with maximum degree  $\Delta$  is a subgraph of some  $\Delta$ -regular bipartite graph. Use this fact to give another proof of König's theorem.

**Exercise 2.** A square matrix  $A = (a_{ij})$  of nonnegative real numbers is called *doubly stochastic* if the entries of each row and each column sum up to 1, that is, for every  $i$  and  $j$ ,

$$\sum_i a_{ij} = \sum_j a_{ij} = 1.$$

A doubly stochastic matrix with all entries in  $\{0, 1\}$  is called a *permutation matrix*. Prove the *Birkhoff-von Neumann theorem*, which states that every doubly stochastic matrix is a convex combination of permutation matrices.