Probabilistic methods in combinatorics

Homework assignment #1

Due date: Monday, April 8, 2019

Problem 1. Let $\mathcal{F}$ be a finite collection of binary strings of finite lengths such that no member of $\mathcal{F}$ is a prefix of another. Let N_i denote the number of strings of length i in $\mathcal{F}$. Prove that

$$\sum_{i=1}^{\infty} \frac{N_i}{2^i} \leqslant 1.$$

Problem 2. Let $k \geqslant 1$ and suppose that $\mathcal{A}$ is a collection of subsets of $[n]$ that does not contain a chain o $k+1$ sets. That is, there are no $A_1, \ldots, A_{k+1} \in \mathcal{A}$ such that $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Show that $\mathcal{A}$ can have at most as many elements as the largest union of k levels of the Boolean lattice, that is,

$$|\mathcal{A}| \leqslant \max \left\{ \sum_{i \in I} \binom{n}{i} : I \subseteq [n] \text{ and } |I| = k \right\}.$$

Problem 3. Prove that an arbitrary set of ten points in the plane can be covered by a family of pairwise disjoint unit disks.

Problem 4. Prove that every set of n nonzero integers contains two disjoint sum-free subsets B_1 and B_2 such that $|B_1| + |B_2| > 2n/3$.

Problem 5. Suppose that G is a graph with no isolated vertices. Show that there exist disjoint independent sets I and J such that

$$|I| + |J| \geqslant \sum_{v \in V(G)} \frac{2}{\deg_G v + 1}.$$

Problem 6. Show that there is a positive constant δ such that the following holds. Suppose that $a_1, \ldots, a_n \in \mathbb{R}$ satisfy $a_1^2 + \cdots + a_n^2 = 1$ and that $\varepsilon_1, \ldots, \varepsilon_n$ are independent random variables with $\Pr(\varepsilon_i = 1) = \Pr(\varepsilon_i = -1) = 1/2$ for every i. Then

$$\Pr\big(|\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n| \leqslant 1\big) > \delta.$$

Please do NOT submit written solutions to the following exercises:

Exercise 1. Modify the argument we used to show that every n-uniform hypergraph with *fewer* than 2^{n-1} edges is 2-colourable so that it yields the stronger assertion that every n-uniform hypergraph with *at most* 2^{n-1} edges is 2-colourable.

Exercise 2. Show that the Ramsey number $R(4,k)$ satisfies $R(4,k) \geqslant c(k/\log k)^2$ for some constant $c > 0$.

Exercise 3. Deduce from the theorem of Ajtai–Komlós–Szemerédi that $R(3,k) \leqslant Ck^2/\log k$ for some constant C.