Probabilistic Methods in Combinatorics

Homework assignment #2

Due date: Sunday, December 14, 2014

Problem 1. Let X be a random variable taking nonnegative integer values. Prove that

$$\Pr(X = 0) \leqslant \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$

Problem 2. Show that there is a positive constant δ such that the following holds. Suppose that $a_1, \ldots, a_n \in \mathbb{R}$ satisfy $a_1^2 + \ldots + a_n^2 = 1$ and that $\varepsilon_1, \ldots, \varepsilon_n$ are independent random variables with $\Pr(\varepsilon_i = 1) = \Pr(\varepsilon_i = -1) = 1/2$ for every i. Then

$$\Pr\big(|\varepsilon_1 a_1 + \ldots + \varepsilon_n a_n| \leqslant 1\big) > \delta.$$

Problem 3. Let $v_1, \ldots, v_n$ be two-dimensional vectors whose coordinates are positive integers not exceeding $2^{n/2}/(10\sqrt{n})$. Prove that there are two disjoint nonempty sets $I, J \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

Problem 4. Let G be a graph with maximum degree D and let $V_1 \cup \ldots \cup V_s$ be a partition of $V(G)$ into s pairwise disjoint sets such that $|V_i| \geqslant 2eD$ for each i. Prove that there is an independent set of G containing precisely one vertex from each V_i.

Problem 5. A coloring f of the vertices of a graph G is *nonrepetitive* if there is no simple path $v_1 \ldots v_{2r}$ in G with $f(v_i) = f(v_{r+i})$ for each i. Prove that there is a constant C such that every simple graph G with maximum degree D admits a nonrepetitive coloring with CD^2 colors.

Problem 6. A 1-subdivision of a graph G with n vertices and m edges is the (bipartite) graph with $n + m$ vertices and $2m$ edges obtained from G by replacing each of its edges with a path of length 2. Prove that every n-vertex graph with εn^2 edges contains a 1-subdivision of a complete graph with $\lfloor \varepsilon^{3/2} n^{1/2} \rfloor$ vertices.