Probabilistic Methods in Combinatorics

Homework assignment #4

Due date: Sunday, January 25, 2015

Problem 1. Prove that there exists an absolute constant c such that for every integer $n \geqslant 2$, there is an interval I_n of at most $c\sqrt{n}/\log n$ consecutive integers satisfying

$$\Pr\Big(\chi\big(G(n,1/2)\big) \in I_n\Big) \geqslant 0.99.$$

Problem 2. Let σ be a uniformly chosen random permutation of $\{1,\dots,n\}$ and let $\mathrm{LIS}(\sigma)$ be the length of a longest increasing subsequence of σ. Show that there exist positive constants c and C such that

$$\lim_{n\to\infty} \Pr\left(c\sqrt{n} \leqslant \mathrm{LIS}(\sigma) \leqslant C\sqrt{n}\right) = 1.$$

Problem 3. The length of a longest common subsequence of two sequences x and y, denoted $\mathrm{LCS}(x,y)$, is the largest integer ℓ such that there are $i_1 < \ldots < i_\ell$ and $j_1 < \ldots < j_\ell$ for which $x_{i_k} = y_{j_k}$ for all $k \in \{1,\dots,\ell\}$.

Let $M\colon \mathbb{N} \to \mathbb{N}$ be arbitrary and suppose that $x = x(n)$ and $y = y(n)$ are two independent uniformly chosen $\{1,\dots,M(n)\}$-valued sequences of length n. Prove that there exists an $\ell\colon \mathbb{N} \to \mathbb{N}$ such that for every $\omega\colon \mathbb{N} \to \mathbb{R}$ satisfying $\omega(n) \to \infty$,

$$\lim_{n\to\infty} \Pr\left(\big|\mathrm{LCS}(x,y) - \ell(n)\big| \leqslant \omega(n)\sqrt{\ell(n)}\right) = 1.$$

Problem 4. Prove that for every positive ε, there is some n_0 such that for every $n > n_0$, there is an n-vertex graph which contains *every* graph on $\lfloor (2-\varepsilon)\log_2 n \rfloor$ vertices as an *induced* subgraph.

Problem 5. Find a threshold function for the following property: $G(n,p)$ contains at least $n/8$ (pairwise) vertex-disjoint copies of K_4.

Problem 6. Let X be a square-integrable real-valued random variable, let $\mu = \mathbb{E}[X]$, and let m be the median of X, that is, a number for which $\Pr(X \geqslant m) \geqslant 1/2$ and $\Pr(X \leqslant m) \geqslant 1/2$. Prove that $|\mu - m| \leqslant \sqrt{\mathrm{Var}(X)}$. (Hint: Use Jensen's inequality.)