Advanced Topics in Computational and Combinatorial Geometry
Assignment 1

Due: March 28, 2016

Problem 1

Davenport-Schinzel sequences of order 2 and triangulations: Let P be any convex polygon with n vertices. A triangulation of P is a collection of $n-3$ non-intersecting chords connecting pairs of vertices of P and partitioning P into $n-2$ triangles. Set up a correspondence between such triangulations and $DS(n-1, 2)$ sequences, as follows. Number the vertices $1, 2, \ldots, n$ in their order along ∂P. Let T be a given triangulation. Include in T the edges of P too. For each vertex i, let $T(i)$ be the sequence of vertices $j < i$ connected to i in T and arranged in *decreasing* order, and let U_T be the concatenation of $T(2), T(3), \ldots, T(n)$.

(a) Show that U_T is a $DS(n-1, 2)$ sequence of maximum length.

(b) Show that any $DS(n-1, 2)$-sequence of maximum length can be realized in this manner, perhaps with an appropriate renumbering of its symbols.

(c) Use (a) and (b) to show that the number of different $DS(n, 2)$ sequences of maximum length is $\frac{1}{n-1}\binom{2n-4}{n-2}$ (where two sequences are different if one cannot obtain one sequence from the other by renumbering its symbols).
(**Note:** Obviously, you have to show that this is the number $F(n)$ of triangulations of P. Show it, e.g., by deriving a recurrence relation on $F(n)$ and solving it inductively.)

Problem 2

(a) Show that $\lambda_3(n) \geq 5n - 8$.

(b) Show that the lower bound in (a) can be realized as the lower envelope sequence of n segments.

Problem 3

Let F be a collection of n partially-defined and continuous functions over the reals. Suppose that F is the disjoint union of c subcollections $F_1, \ldots, F_c$, such that (i) Any pair of functions in F_i intersect in at most s_i points, for $i = 1, \ldots, c$. (ii) Any pair of functions of F intersect in at most s points. Here c, s, and the s_i's are all constants.

(a) Show that the complexity of the lower envelope of F is $O(\lambda_{q+2}(n))$, where $q = \max_{i=1}^{c} s_i$.

(b) Show that the envelope can be computed in time $O(\lambda_{q+1}(n) \log n)$.

(c) As a corollary, what is the complexity of the lower envelope of a collection of n line segments and unit-radius circles? (For the purpose of the lower envelope, each circle can be replaced by its lower semi-circle.)

Problem 4

The transversal region of triangles: Let $\mathcal{T} = \{T_1, \ldots, T_n\}$ be a collection of n triangles in the plane. A line ℓ is called a *transversal* of $\mathcal{T}$ if it intersects all triangles of $\mathcal{T}$. The *transversal region* of $\mathcal{T}$ is the set of all points dual to the transversal lines of $\mathcal{T}$.

(a) What is the shape of the transversal region if $\mathcal{T}$ contains just one triangle?

(b) What is the shape of the stabbing region for a general $\mathcal{T}$?

(c) Show that the complexity of the stabbing region is $O(n\alpha(n))$.

Problem 5 (bonus problem)

Getting wild with Ackermann:
Prove property (C4) for the function $C_k(m)$ defined in class:

$$A_{k-1}(m) \leq C_k(m) \leq A_k(m+3)$$

for $k \geq 4$, $m \geq 1$.