Advanced Topics in Computational and Combinatorial Geometry
Assignment 2

Due: April 11, 2016

Problem 1

Show that a single cell in an arrangement of n rays in the plane has complexity $O(n)$. Use this to show that the complexity of the *outer zone* of any closed convex curve γ in an arrangement of n lines in the plane is $O(n)$. (The outer zone is the portion of the zone that lies outside γ.) What can you say about the complexity of the inner zone (the portion of the zone inside γ)?

Problem 2

Show that the zone of a line in an arrangement of n unit circles in the plane has complexity $O(n\alpha(n))$. What happens when the circles have arbitrary radii? (**Hint:** For the first part, consider separately the top and bottom portions of each circle.)

Problem 3

Let L be a set of n lines in the plane. Define the *separation distance* $d_L(p, q)$ between a pair of points $p, q \in \mathbb{R}^2$ to be the number of lines of L crossed by the segment pq. Show that d_L satisfies the triangle inequality.

(a) Show that for any point x and integer $k \leq n$ there are $\Omega(k^2)$ vertices of $\mathcal{A}(L)$ at separation distance $\leq k$ from x.

(b) Conclude that, for any set P of m points in the plane there are always two points $p, q \in P$ at separation distance at most $O(n/\sqrt{m})$ apart. (**Hint:** Use a packing argument involving the vertices of $\mathcal{A}(L)$.)

(c) Use (b) to show that, for a given set P of m points, the minimum spanning tree T of P under the separation distance has total weight $O(n\sqrt{m})$. Note that this means that, on average, each line of L crosses only $O(\sqrt{m})$ edges of T. (**Hint:** Construct tree edges iteratively, using (b) and deleting points of P.)

Problem 4

Dynamic Voronoi diagram. Let $P = \{p_1(t), \ldots, p_n(t)\}$ be a set of n points moving in the plane. Assume that for each $i = 1, \ldots, n$, each coordinate of $p_i(t)$ is given as a polynomial in t of degree at most k. Let $Vor(t)$ denote the Voronoi diagram of P at time t. The combinatorial structure of $Vor(t)$ is the description of the diagram as a planar map, ignoring the specific coordinates of its vertices and equations of its edges. Thus, the representation of a single Voronoi cell $V(p)$ at time t is simply the circular list of its neighbors (those points of P that form neighboring cells at time t), in the counterclockwise order that they appear along the boundary of $V(p)$.

Show that the maximum possible number of changes in the combinatorial structure of $Vor(t)$ over time is $O(n^2\lambda_s(n))$, where s is a constant depending on k (give an upper bound on s). (**Hint:** Fix a pair of points p_i, p_j, and write the sequence of points of P that form, over time, one of the Voronoi vertices on the Voronoi edge between $V(p_i)$ and $V(p_j)$ (when this edge exists at all). Treat each vertex separately and be careful about the 'side' of the vertex along the bisector.)