Assignment 3
Advanced Topics in Computational Geometry

Due: May 9, 2016

The topic in Problem 1(c) has not yet been covered in class—I gave you extra time for this assignment. The other problems can already be approached.

Problem 1

Let L be a set of n lines in the plane. For each triangle Δ, let L_Δ denote the set of lines of L that cross Δ.

(a) Show that the number of different sets L_Δ is $O(n^6)$.

(b) Show that the number of triples u, v, w of vertices of $\mathcal{A}(L)$ such that no line crosses the triangle uvw (ignoring lines that touch it at u, v, or w) is $O(n^3)$, and that this bound is tight in the worst case. (**Hint:** The zone theorem may be useful here.)

(c) Using (b), show that the number of triples u, v, w of vertices of $\mathcal{A}(L)$ such that the triangle uvw is crossed by at most k lines of L, is $O(n^3k^3)$.

Problem 2

Let P be a set of n points on the line, and let R be a random sample of points of P, such that each point is chosen independently with probability r/n (so the expected size of R is r). Let $\varphi(R)$ denote the largest cardinality of $I \cap P$, over all intervals I that do not contain any point of R. Show that, with high probability, $\varphi(R) \leq \frac{cn}{r} \log r$, for some sufficiently large constant c. (Show that the probability of the complementary event is at most $1/r^k$, where k depends on the constant c.)

(**Hint:** Fix an interval I for which $|I \cap P| \geq \frac{cn}{r} \log r$, and estimate the probability that it does not contain any point of R. Define the collection of such intervals carefully and use the probability union bound.)

Problem 3

(a) Let S be a set of n non-vertical line segments in the plane (in general position). We insert the elements of S one by one in a random order, and maintain the lower envelope of S as we go, so that after each insertion we update the envelope, to reflect the presence of the new segment.

Show that the expected number of vertices that the algorithm generates is $O(n\alpha(n))$.

(b) Design an efficient algorithm that computes the lower envelope, using the above randomized incremental insertion, so that its expected running time is $O(n\alpha(n)\log n)$. (**Hint:** Maintain the vertical decomposition of the portion of the plane below the envelope into a collection of vertical semi-unbounded trapezoids.)

Note: Do not cite what we did in class. This is a special simple variant of the general case, and the exercise is to solve it "from scratch" in a (somewhat) simpler manner.

Problem 4

(a) Use the zone bound to construct an arrangement of a set L of n lines in the plane, in the following deterministic incremental manner: Insert the lines of L one by one in any order. When a new line ℓ is inserted, find the leftmost face f_1 of the current arrangement that ℓ intersects, scan the boundary of f_1 to find the (unique) intersection point of ℓ with the boundary, cross through this point to the next face f_2, and repeat the procedure for f_2, and keep going till the rightmost face crossed by ℓ is reached.

Show that the algorithm takes $\Theta(n^2)$ time.

(b) Derive a generalization of this algorithm to the case where L is a collection of n graphs of totally defined continuous functions, such that each pair of them intersect in at most s points. What is the complexity of the resulting algorithm?

(c) Same as (b), when L is a collection of n graphs of partially defined continuous functions, such that each pair of them intersect in at most s points.

Assume in (b) and (c) a suitable model of computation.