Assignment 4
Advanced Topics in Computational Geometry

Due: May 30, 2016

Problem 1

Minkowski sums: Let $A_1, \ldots, A_m$ be a collection of m pairwise-disjoint convex polygons in the plane, so that A_i has n_i edges, for a total of $n = \sum_{i=1}^{m} n_i$ edges. Let B be a convex polygon with k edges. Assume general position of the A_i's and B. The *Minkowski sum* of A_i and B, for $i = 1, \ldots, m$, is defined as

$$K_i = A_i \oplus B = \{x + y \mid x \in A_i, y \in B\}.$$

(a) Show that, for any (generic) direction θ, the extreme point of K_i in direction θ (this is the point where a line whose outward normal has direction θ supports K_i) is $x_\theta + y_\theta$, where x_θ (resp., y_θ) is the extreme point of A_i (resp., B_i) in direction θ.

(b) Show that each K_i is a polygon with $n_i + k$ edges.

(c) Show that the boundaries of any pair of the polygons K_i, K_j intersect in at most 2 points. (**Hint:** Use (a) to show that K_i and K_j have at most two common outer tangents.)

(d) Give an efficient algorithm for computing the union $K = \bigcup_{i=1}^{m} K_i$ (use a divide-and-conquer approach).

Problem 2

(a) Let L be a set of n lines in the plane (in general position). Show that $\sum_c |c|^2 = \Theta(n^2)$, where the sum is over all cells c of $\mathcal{A}(L)$, and where $|c|$ is the complexity of c (e.g., number of vertices). (**Hint:** Use the zone theorem.)

(b) Let H be a set of n planes in three dimensions. Show that $\sum_c |c|^2 = \Theta(n^3)$, where the sum is over all cells c of $\mathcal{A}(H)$, and where $|c|$ is the total complexity of c (i.e., number of vertices, edges, and faces of c). (**Hint:** Use the zone theorem.)

(c) The analysis does not work in four or higher dimensions. Explain why, and show that for any dimension d we have instead $\sum_c |c| \cdot |c|_{d-1} = \Theta(n^d)$, where $|c|_{d-1}$ is the number of $(d-1)$-dimensional faces (facets) bounding c.

(d) Using (a), show that the total complexity of m arbitrary faces of $\mathcal{A}(L)$ is $O(m^{1/2}n)$. (**Hint:** Cauchy-Schwarz!)

Problem 3

Let L be a set of n lines in the plane (in general position). Call a face f of $\mathcal{A}(L)$ *balanced* if the highest vertex of f (in the y-direction) is not the leftmost or rightmost vertex of f; otherwise f is *tilted*.

Using the Clarkson-Shor technique, show that the number of balanced faces at level at most k is $O(k^2)$. Show also that the bound is tight in the worst case.

Problem 4

Let $\mathcal{F}$ be a collection of n bivariate functions of constant description complexity. Using the Clarkson-Shor technique, show that if the functions are inserted one by one in a random order, and the envelope is updated after each function is inserted, then (a) the expected number of vertices (points where three functions intersect on the current envelope) that are generated by the algorithm is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, and (b) that the expected sum of the *weights* of these vertices is also $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the weight of a vertex is the number of function graphs that pass below it. Give an example of a bad (non-random) insertion order for which $\Theta(n^3)$ vertices are generated. (Note: This is not an algorithmic question; we do not care how exactly the envelope is maintained, but require that we form the new, correct version of the envelope after each function is inserted.)

Problem 5

Let $\mathcal{F}$ be a collection of n disks in three dimensions, none of which is vertical. Regard these disks as the graphs of n partially defined bivariate functions, and analyze the complexity of the lower envelope of these functions, applying a (simpler) variant of the technique shown in class (or any technique of your choice). Give a construction showing that the complexity of the envelope can be $\Omega(n^2)$, even when the disks are disjoint.

Note: Because the disks are "partially defined", the envelope has additional kinds of vertices, such as a point lying on two disks and above the boundary of a third disk, and also a point that lies on the boundary of one disk and above the boundary of another disk. The precise way of thinking about these "fake" vertices is that they are vertices of the xy-projection of the lower envelope (the so-called *minimization diagram* of $\mathcal{F}$).