# The Clarkson-Shor Technique Revisited and Extended*

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**Abstract**

We provide an alternative, simpler and more general derivation of the Clarkson-Shor probabilistic technique [6] and use it to obtain in addition several extensions and new combinatorial bounds.

## 1 Introduction

In abstract and general setting, the Clarkson-Shor technique [6] (originally presented by Clarkson [5]) deals with the following type of problems. Let $S$ be a set of $n$ objects, and $C$ a set of *configurations*, each defined by $d$ objects of $S$, for some constant integer parameter $d$. We are also given a *conflict relationship* between objects $a \in S$ and configurations $c \in C$, where it is assumed that the $d$ objects that define $c$ are not at conflict with it. The *weight* of a configuration $c$ is the number of objects that are in conflict with $c$.

As a concrete example, let $S$ be a set of $n$ lines (the objects) in the plane. A configuration is a vertex of the arrangement $\mathcal{A}(S)$, defined by $d = 2$ lines. A line is at conflict with a vertex if it passes below the vertex. The zero-weight vertices are those that appear on the lower envelope of the lines, and their number is at most $n - 1$. Vertices of weight $k$ belong to the $k$-th level of the arrangement.

Let $C_0(S)$ denote the set of 0-weight configurations, let $C_k(S)$ denote the set of configurations of weight exactly $k$, and let $C_{\le k}(S)$ denote the set of configurations of weight at most $k$, where $k$ is any integer between 0 and $n - d$. Put $N_0(S) = |C_0(S)|$, $N_k(S) = |C_k(S)|$, and $N_{\le k}(S) = |C_{\le k}(S)|$. We also denote by $N_0(n)$ (resp. $N_k(n)$, $N_{\le k}(n)$) the maximum of $N_0(S)$ (resp. of $N_k(S)$, $N_{\le k}(S)$), over all sets $S$ of $n$ objects of this kind.

The Clarkson-Shor technique provides the following upper bound for $N_{\le k}(S)$:

**Theorem 1.1 (Clarkson and Shor [6])**

$$N_{\le k}(n) = O(k^d N_0(n/k)). \tag{1}$$

We derive a somewhat different bound, which can then be manipulated to yield the Clarkson-Shor's bound—see below.

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**Theorem 1.2**

$$N_{\leq k}(S) \leq \frac{\mathbf{E}(N_0(R_p))}{p^d(1-pk)}, \tag{2}$$

*for any $0 < p < 1/k$, where $R_p$ is a random sample of objects from $S$, where each object is chosen independently with probability $p$, and $N_0(R_p)$ is the number of configurations, all of whose defining objects are in $R_p$ and none of whose conflicting objects are in $R_p$.*

**Remark:** Theorem 1.1 can in fact also be stated in terms of expectation: It asserts that $N_{\leq k}(S)$ is upper bounded by $O(k^d)$ times the expected value of $N_0(R'_{n/k})$, where $R'_{n/k}$ is a random sample of $n/k$ objects from $S$, where all such samples are chosen with equal probability.

By now, the Clarkson-Shor technique needs no praise—it has become a cornerstone of many of the developments in computational and combinatorial geometry in the past decade, for example, in the analysis of randomized incremental algorithms [11], and in the derivation of sharp bounds on the complexity of lower envelopes and other substructures in higher-dimensional arrangements [13]. This paper serves two main purposes:

(i) It puts the Clarkson-Shor's technique in a somewhat different context, leading to a simpler proof and perhaps also to a better understanding of the technique.

(ii) It facilitates the extension of the context in which the technique can be applied. We obtain in fact several results which do not seem to be derivable directly by the original technique, even though there exist alternative techniques that one could use for deriving the bounds obtained here.

Our proof technique is a simple extension of the probabilistic proof of the *Crossing Lemma* of Leighton [10] and of Ajtai et al. [2]. (This proof is so elegant that it has achieved a status of 'a proof from the *Book*' in [1].) On one hand, this technique is quite general, but, on the other hand, in almost all applications to date, it has been administered to crossings between edges of graphs drawn in the plane (see [3, 7] for some exceptions). It is our hope that the results reported here would constitute a first step towards a wider range of applications of the technique.

## 2 Analysis

Before proceeding to the proof of Theorem 1.2, we first establish the following more general result.

Let $C'$ be any subset of configurations. Let $X$ be the number of conflicts with the configurations of $C'$. That is,

$$X = |\{(a, c) \mid c \in C', a \text{ is at conflict with } c\}|$$

We then have

$$X \geq |C'| - N_0(S). \tag{3}$$

Indeed, if $|C'| > N_0(S)$ then $C'$ contains at least one configuration $c$ that has at least one conflict, contributing at least 1 to the count $X$. Remove $c$ from $C'$ and repeat this argument, and keep doing so as long as $|C'| > N_0(S)$.

Draw a random sample $R$ of objects from $S$, by drawing each $a \in S$ independently, with the same probability $p$. Let $C'_R$ denote the set of configurations in $C'$ that appear in $R$, that is, those configurations for which all $d$ defining objects are chosen in $R$, and let $X_R$ denote the number of conflicts of these configurations in $R$, that is, the number of pairs $(a, c)$ such that $c \in C'_R$, $a \in R$ and $a$ is at conflict with $c$.

By (3), we have

$$X_R \geq |C'_R| - N_0(R),$$

so the same holds for the expectation of this inequality:

$$\mathbf{E}(X_R) \geq \mathbf{E}(|C'_R|) - \mathbf{E}(N_0(R)).$$

It follows from the construction that

$$\mathbf{E}(|C'_R|) = |C'|p^d \quad \text{and} \quad \mathbf{E}(X_R) = Xp^{d+1}.$$

Hence we obtain

**Theorem 2.1** *For any set $C'$ of configurations, which have a total of $X$ conflicts, and for any probability $0 < p \leq 1$, one has*

$$Xp \geq |C'| - \frac{\mathbf{E}(N_0(R_p))}{p^d}, \tag{4}$$

*where $R_p$ is as in the statement of Theorem 1.2.*

**Proof of Theorem 1.2:** Let us specialize Theorem 2.1 to the case $C' = C_{\leq k}(S)$. In this case we have, by definition, $X \leq kN_{\leq k}(S)$, which implies:

$$(1 - pk)N_{\leq k}(S) \leq \frac{\mathbf{E}(N_0(R_p))}{p^d},$$

and therefore completes the proof. □

**A quick illustration.** Consider the case of lines and vertices mentioned in the introduction. We have $\mathbf{E}(N_0(R_p)) = \mathbf{E}(|R_p| - 1) \leq np$, so

$$N_{\leq k}(S) \leq \frac{np}{p^2(1 - pk)} \leq 4nk,$$

by choosing $p = 1/(2k)$.

**The Clarkson-Shor bound.** Let us turn (2) into the more familiar bound given in Theorem 1.1. Suppose that $N_0(S) \leq A\binom{n}{\gamma}$, for any set $S$ of $n$ objects and for some integer parameter $\gamma \geq 1$ and constant $A > 0$. We have

$$\mathbf{E}\left[\binom{|R_p|}{\gamma}\right] = \binom{n}{\gamma}p^\gamma.$$

Indeed, the left-hand side is the expected number of $\gamma$-tuples of distinct objects in $R_p$. The right-hand side provides an explicit expression for this quantity, observing that there are $\binom{n}{\gamma}$ $\gamma$-tuples of distinct objects in $S$, and that the probability of any of them to materialize in $R_p$ is $p^\gamma$.

Hence (2) becomes

$$N_{\leq k}(S) \leq \frac{A\binom{n}{\gamma}}{p^{d-\gamma}(1 - pk)}.$$

Now choose $p = \beta/k$, for some parameter $\beta \in (0, 1)$, to obtain

$$N_{\leq k}(S) \leq \frac{A\binom{n}{\gamma}k^{d-\gamma}}{\beta^{d-\gamma}(1 - \beta)}.$$

The denominator is maximized when $\beta = (d - \gamma)/(d - \gamma + 1)$, which is easily seen to yield

$$N_{\leq k}(n) \leq A(d - \gamma + 1)e\binom{n}{\gamma}k^{d-\gamma}. \tag{5}$$

This is essentially the Clarkson-Shor result.

## 3 Extensions—Simple Constraints

Recall that the inequality (4) of Theorem 2.1 is fairly general, and does not impose any specific assumptions on the set $C'$. We next apply this theorem to several other problems involving sets of configurations $C'$ whose overall number of conflicts can be upper bounded by some other simple argument, which will lead to various new upper bounds on the size of such sets $C'$. In the following section, we will extend the technique further, by considering conflicts between a configuration and *several* objects.

**Configurations for which no object is at conflict with many.** Here we consider an 'inverse problem', where we wish to bound the maximum possible size of a set $C^*_{\le k}$ of configurations so that no object is at conflict with more than $k$ configurations in $C^*_{\le k}$.

For such sets we have trivially $X \le nk$. Assume also, as above, that $N_0(n) \le A\binom{n}{\gamma}$, for some integer $\gamma \ge 1$ and constant $A > 0$. Then (4) becomes

$$|C^*_{\le k}| \le nkp + \frac{A\binom{n}{\gamma}}{p^{d-\gamma}}.$$

Choose

$$p = \left(\frac{A\binom{n}{\gamma}}{nk}\right)^{1/(d-\gamma+1)},$$

which makes sense only when $k \ge A\binom{n}{\gamma}/n$. Assuming this to be the case, we have

$$|C^*_{\le k}| = O\left(n^{d/(d-\gamma+1)}k^{(d-\gamma)/(d-\gamma+1)}\right).$$

If $k < A\binom{n}{\gamma}/n$, choose $p = 1$ to obtain $|C^*_{\le k}| \le 2A\binom{n}{\gamma}$. We thus obtain:

**Theorem 3.1** *Assume the above abstract setup of objects, configurations, and conflicts, and suppose that the number of configurations with no conflicts in any set of $n$ objects is $O(n^\gamma)$. Let $1 \le k \le n$ be a given parameter. Then the maximum cardinality of any set of configurations with the property that no object is at conflict with more than $k$ of them is*

$$O\left(n^\gamma + n^{d/(d-\gamma+1)}k^{(d-\gamma)/(d-\gamma+1)}\right). \tag{6}$$

**Examples.** (1) Suppose that, as in the introduction, the objects are $n$ lines in the plane, configurations are vertices of their arrangement, and a vertex is at conflict with a line if the vertex lies above the line. In this case we have $d = 2$, $\gamma = 1$, and we obtain

**Corollary 3.2** *The maximum number of vertices in an arrangement of $n$ lines in the plane, such that none of the lines passes below more than $k$ of them, is $O(nk^{1/2})$. This bound is tight in the worst case. Dually, the maximum number of lines connecting pairs of points in an $n$-element point set in the plane, such that none of the given points lies below more than $k$ connecting lines, is $O(nk^{1/2})$, which again is worst-case tight.*

The lower bound is obtained as follows. Take $n/k^{1/2}$ lines, all appearing along their lower envelope. replace each line by a bundle of $k^{1/2}$ parallel lines sufficiently close to each other. Each vertex of the lower envelope of the original lines is replaced by $k$ new vertices. Collecting all these vertices, we obtain a set of $nk^{1/2}$ vertices, and it is clear that no line passes below more than $2k$ of them.

(2) A similar problem can be stated and analyzed for hyperplanes in $\mathbb{R}^d$, where the configurations are vertices in an arrangement of $n$ hyperplanes, with the parameter $d$ equal to the dimension, and $\gamma = \lfloor d/2 \rfloor$:

**Corollary 3.3** *The maximum number of vertices in an arrangement of $n$ hyperplanes in $\mathbb{R}^d$, for which no hyperplane passes below more than $k$ of them, is*

$$O\left(n^{\lfloor d/2\rfloor} + n^{d/(\lceil d/2\rceil+1)}k^{\lceil d/2\rceil/(\lceil d/2\rceil+1)}\right),$$

*and this bound is tight in the worst case. The same bound holds for the maximum number of hyperplanes spanned by a set of $n$ points in $\mathbb{R}^d$ so that no point lies below more than $k$ of them.*

The lower bound follows from a construction similar to that for the planar case, which is based on the upper bound theorem for convex polytopes.

In $d = 3$ dimensions, we get the bound $O(nk^{2/3})$. By using a standard lifting transform from the plane to three dimensions [9] and by specializing the preceding corollary to $d = 3$, we also obtain:

**Corollary 3.4** *The maximum number of circles spanned by $n$ points in the plane, such that none of the given points lies in more than $k$ circles, is $O(nk^{2/3})$, and this bound is tight in the worst case.*

## 4 Extensions—More General Conflicts

So far, we have only considered conflicts, each involving one configuration and one single object, but the technique is sufficiently powerful to allow us to consider more elaborate types of conflicts, each involving one configuration and *several* objects. We illustrate this in two examples:

In the first example, the objects are lines in the plane, the configurations are triangles bounded by triples of the lines, and a triangle $\Delta$ is at conflict with two other lines $\ell_1, \ell_2$ if the vertex $\ell_1 \cap \ell_2$ lies in the interior of $\Delta$.

In the second example, the objects are points in 3-space, the configurations are triangles spanned by triples of the points, and a triangle $\Delta$ is at conflict with two other points $u, v$ iff the segment $uv$ crosses the relative interior of $\Delta$.

Handling conflicts of this kind can be done by a straightforward modification of the method of Section 2. Specifically, suppose that a conflict involves one configuration and $b$ objects. Starting from the inequality $X \geq |C'| - N_0(S)$, which clearly holds in this case too, and passing to a random sample $R_p$ as above, we have

$$\mathbf{E}(|C'_R|) = |C'|p^d \quad \text{and} \quad \mathbf{E}(X_R) = Xp^{d+b},$$

so we obtain:

**Theorem 4.1** *For any set $C'$ of configurations, which have a total of $X$ conflicts, each involving one configuration and $b$ objects, and for any probability $0 < p \leq 1$, one has*

$$Xp^b \geq |C'| - \frac{\mathbf{E}(N_0(R_p))}{p^d}, \tag{7}$$

*where $R_p$ is as in the statement of Theorem 1.2.*

We next apply this theorem to the examples mentioned above.

### 4.1 Triangles and vertices in a line arrangement

**Theorem 4.2** *In an arrangement of $n$ lines in the plane in general position, there are at most $O(n^2k^{1/2})$ triangles whose edges lie on three of the given lines and which contain at most $k$ vertices of the arrangement in their interiors. This bound is tight in the worst case.*

**Proof:** Let $L$ be the given set of lines. Here the objects are the lines of $L$ and the configurations are triangles bounded by triples of lines in $L$. Conflicts are more involved: A triangle is at conflict with a vertex of $\mathcal{A}(L)$ if it contains the vertex in its interior. Thus a conflict is defined in terms of 5 lines: three defining the triangle and two the vertex.
**Claim:** The number $N_0(L)$ of triangles spanned by three lines of $L$ and containing no vertex of $\mathcal{A}(L)$ in their interior is $O(n^2)$.

Indeed, let $T_0$ be the set of triangles that are bounded by three lines of $L$ and do not contain any vertex in their interior. The number of triangles in $T_0$ that are not crossed by any line of $L$ is clearly $O(n^2)$—they are faces of the arrangement.

Suppose then that a triangle $\Delta$ of $T_0$ is crossed by at least one line $\ell$ but contains no vertex in its interior. Let $a, b, c$ denote the vertices of $\Delta$. If there exist lines that cross $\partial\Delta$ at the edges $ab, ac$, choose from among them the line whose intersections with these edges are further away from $a$, and denote it by $\ell_a$ (since all these lines do not cross inside $\Delta$, $\ell_a$ is well-defined). Define $\ell_b$, $\ell_c$ in an analogous fashion, when they exist. See Figure 1(a). At least one of these lines must exist; assume, without loss of generality, that $\ell_a$ exists.

We charge $\Delta$ to an intersection point $v$ of $\ell_a$ with $\partial\Delta$, say with the line $\ell_0$ that contains $ab$. We claim that there can exist at most one other triangle $\Delta'$ in $T_0$ that is bounded by $\ell_0$, lies on the same side of $\ell_0$ as $\Delta$, contains $v$ on its boundary, and charges $v$. Indeed, suppose to the contrary that two such triangles $\Delta', \Delta''$ exist. The intersection $\Delta \cap \Delta'$ is a convex polygon that contains $v$ on its boundary and is crossed by $\ell = \ell_a$. Hence $\ell$ must intersect the boundary of this region at a second point $w$. Without loss of generality, assume that $w$ lies on $\partial\Delta$. Since $w$ is a vertex of $\mathcal{A}(L)$, it cannot lie in the interior of $\Delta'$. Hence, $\Delta$ and $\Delta'$ must share the vertex $a$, and their edges incident to $a$ overlap in pairs. The same holds for $\Delta$ and $\Delta''$. See Figure 1(b).

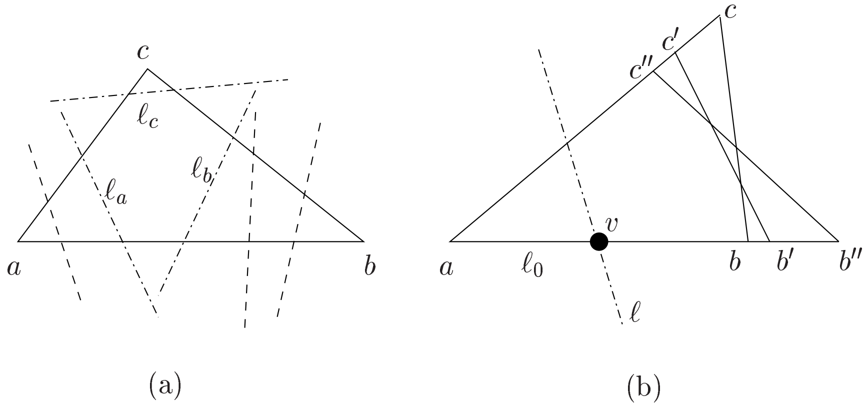


Figure 1: (a) The lines $\ell_a, \ell_b, \ell_c$ for the triangle $\Delta = abc$. (b) A triple charging of a vertex $v$ is impossible.

Consider the three edges of $\Delta, \Delta', \Delta''$ opposite to $a$, and denote them by $bc, b'c'$, and $b''c''$, respectively. Suppose that there is a pair of these edges, say, $bc$ and $b'c'$, that do not cross each other, with $b'c'$ closer to $a$ than $bc$. Then the line $\ell_a$ for the triangle $\Delta$ must be either the line containing $b'c'$ or a line that lies further away from $a$. In either case, it cannot intersect $ab$ at $v$, contrary to assumption. Hence we may assume that all three pairs among $bc, b'c', b''c''$ cross each other, in which case at least one of the three intersection points must lie inside the triangle bounded by the third edge, again a contradiction.

This implies that the number of triangles in question is at most proportional to the number of vertices $v$ of the arrangement, so their number is $O(n^2)$. This completes the proof of the claim.

Let $N_{\le k}(L)$ denote the number of triangles with weight at most $k$, and let $X$ denote the total number of conflicts that they have, which is clearly upper bounded by $X \le kN_{\le k}(L)$. Using Theorem 4.1, we thus obtain

$$kp^2 N_{\le k}(L) \ge N_{\le k}(L) - \frac{\mathbf{E}(N_0(R_p))}{p^3},$$

so

$$N_{\le k}(L) \le \frac{\mathbf{E}(N_0(R_p))}{p^3(1-kp^2)}.$$

Choosing $p = \beta/k^{1/2}$, for an appropriate $\beta < 1$, and using the fact that $N_0(n) = O(n^2)$, the upper bound of the theorem follows readily. The lower bound is obtained for an arrangement consisting of $n/3$ equally-spaced horizontal lines, $n/3$ equally-spaced vertical lines, which together form part of the integer grid, and $n/3$ additional equally-spaced lines of slope 1, passing very near the grid points formed by the first two subfamilies. We leave the easy verification of the lower bound to the reader. □

**Theorem 4.3** *In an arrangement of $n$ lines in the plane, there are at most $O(n^2k^{1/3})$ triangles whose edges lie on three of the given lines and for which no vertex of the arrangement is contained in the interiors of more than $k$ of the triangles. This bound is tight in the worst case.*

**Proof:** Here $X \le k\binom{n}{2}$, so, using Theorem 4.1, the number of such triangles is at most $O(n^2kp^2 + n^2/p)$. The theorem follows by choosing $p = 1/k^{1/3}$. The lower bound can be obtained from the same construction used in the preceding proof. □

**Remark:** The claim in the proof of Theorem 4.2, concerning triangles that contain no vertex in their interior, does not seem to extend to vertical trapezoids in an arrangement of lines.

## 4.2 Triangles and crossing segments in a 3-dimensional point set

**Theorem 4.4** *Given a set of $n$ points in $\mathbb{R}^3$, the maximum number of triangles spanned by the points of $S$ that are crossed by at most $k$ segments connecting pairs of points in $S$ is $O(n^2k^{1/2})$.*

**Proof:** Let $T_{\le k}$ denote the set of these triangles, and let $X$ denote the number of conflicts between triangles in $T_{\le k}$ and segments. We have $X \le k|T_{\le k}(S)|$. By the results of [7, 15], we have $N_0(S) = O(n^2)$. Hence, by Theorem 4.1, we have

$$kp^2|T_{\le k}(S)| \ge |T_{\le k}(S)| - \frac{\mathbf{E}(N_0(R_p))}{p^3},$$

so

$$|T_{\le k}(S)| \le \frac{\mathbf{E}(N_0(R_p))}{p^3(1-kp^2)}.$$

Choosing $p = \beta/k^{1/2}$, for an appropriate $\beta < 1$, and using the fact that $N_0(n) = O(n^2)$, the theorem follows readily. □

**Theorem 4.5** *Given a set of $n$ points in $\mathbb{R}^3$, the maximum number of triangles spanned by the points of $S$, so that no segment connecting a pair of points of $S$ crosses more than $k$ of them, is $O(n^2k^{1/3})$.*

**Proof:** Here we have $X \le k\binom{n}{2}$, so the proof proceeds as in the proof of Theorem 4.3. □

**Corollary 4.6** *The number of halving triangles in an $n$-element point set in 3-space is $O(n^{8/3})$.*

**Proof:** By Lovász lemma (see [4, 15]), any segment can cross at most $O(n^2)$ halving triangles, so the bound follows by substituting $k = O(n^2)$ in Theorem 4.5. □

Note that this bound is weaker than the best known bound $O(n^{5/2})$ [15].

**Corollary 4.7** *The maximum number of distinct triangles that lie on the boundaries of $k$ convex polytopes spanned by a set of $n$ points in 3-space is $O(n^2k^{1/3})$.*

**Proof:** Clearly, no segment crosses more than $2k$ of these triangles, so the bound is an immediate application of Theorem 4.5. □

This result has been obtained by Aronov and Dey [3] using a more involved argument. A simple alternative proof is given in [14].

## 5 Discussion

Clearly, this paper only scratches the surface of the realm of applications of this (extended) technique. For example, in the original setup of the Crossing Lemma of [2, 10], a conflict occurs between two configurations (a conflict is a crossing between two edges, that is, between two configurations), a situation that we haven't considered at all here, but one that should be amenable to the new technique just as the other cases studied here.

We believe that the ideas developed here will have additional applications. For example, our next planned step in this research is to find algorithmic applications for the new bounds, extending similar applications of the standard Clarkson-Shor's bounds, e.g., to the analysis of randomized incremental algorithms.

We end the paper by presenting an alternative interpretation of the analysis employed in this paper. Let $S$ be a set of objects, and let $A$ be a subset of $S^q$, for some $q$. Put $d_A = q$, and refer to it as the *dimension* of $A$. For any $R \subseteq S$, let $A_R = A \cap R^q$.

Suppose now that we have a finite collection $A_1, \ldots, A_\nu$ of such sets of ordered tuples, possibly with different dimensions $d_{A_i}$, with corresponding (positive or negative) constants $c_1, \ldots, c_\nu$, so that the linear relation

$$\sum_{i=1}^{\nu} c_{A_i} |(A_i)_R| \le f(|R|)$$

holds for any $R \subseteq S$, where $f(\cdot)$ is some function of $|R|$.

Then, for any $p \in (0, 1)$, we have

$$\sum_{i=1}^{\nu} c_i p^{d_{A_i}} |A_i| \le \mathbf{E}[f(|R_p|)]. \tag{8}$$

The proof of the Crossing Lemma of [2, 10] is an instance of this observation, using three sets $A_1, A_2, A_3$ of respective dimensions $1, 2, 4$, where $A_1$ is $S$, the set of vertices of the given graph, $A_2$

is the set of its edges, and $A_3$ is the set of edge crossings (each represented by the quadruple of the vertices incident to the pair of crossing edges). The linear relation is $|A_3| - |A_2| + 3|A_1| \geq 0$, and (8), with an appropriate choice of $p$, yields the lemma.

The derivation of, say, Theorem 2.1 can also be interpreted as an application of (8) to the inequality (3), as the reader can easily verify. This interpretation applies also to other theorems derived in this paper.

This view of the analysis presented here shows that it is indeed strongly related to the original Crossing Lemma.

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