(B6) $2^{B_k(m)+2} \le A_k(m+3)$, for $k \ge 1$, $m \ge 0$.

(B7) $A_{k-1}(m) \le B_k(m) \le 2^{B_k(m)} \le A_k(m+3)$, for $k \ge 4$, $m \ge 1$, so that the sequences of functions B_k and A_k have the same asymptotic order of growth.

Next we define another sequence of functions $\{C_k\}$ by putting $C_k(m) = 2^{B_k(m)}$, for all k and m. An explicit recursive definition of these functions, for positive k and m, is

$$\begin{aligned} C_1(m) &= 1, & m \ge 1, \\ C_k(1) &= 2C_{k-1}(2), & k \ge 2, \\ C_k(m) &= C_k(m-1) \cdot C_{k-1}(C_k(m-1)), & k \ge 2, m \ge 2. \end{aligned} \quad (2.4)$$

It follows easily from the preceding analysis that

(C1) $C_2(m) = 2$, for $m \ge 0$.

(C2) $C_3(m) = 2^{m+1}$, for $m \ge 0$.

(C3) $C_4(m) \ge 2^{2^{\cdot^{\cdot^{2}}}}$, with $m+1$ 2's in the exponential tower.

(C4) $A_{k-1}(m) \le C_k(m) \le A_k(m+3)$, for $k \ge 4$, $m \ge 1$, so that the growth of the sequences of functions $\{C_k\}$ and $\{A_k\}$ are also of the same asymptotic order of magnitude.

In what follows we will often use the shorthand notations

$$\bar{\alpha} = C_k(m-1), \quad \bar{\beta} = C_{k-1}(C_k(m-1)) = C_{k-1}(\bar{\alpha})$$

and $\bar{\gamma} = C_k(m) = \bar{\alpha} \cdot \bar{\beta}$ (by definition).

2.3.2 *Generation of superlinear-size sequences*

For each $k, m \ge 1$, the sequence $S(k,m)$ that we are going to construct will satisfy the following two properties:

(1) $S(k,m)$ is composed of $N_k(m) = m \cdot C_k(m)$ distinct symbols. (These symbols are named (d,γ), for $d = 1, \ldots, m$, $\gamma = 1, \ldots, \bar{\gamma}$, and are ordered in lexicographical order, so that $(d,\gamma) < (d',\gamma')$ if $\gamma < \gamma'$ or $\gamma = \gamma'$ and $d < d'$.)

(2) $S(k,m)$ contains $C_k(m)$ *fans* of size m, where each fan is a contiguous subsequence of the form

$$\langle (1,\gamma)\,(2,\gamma) \cdots (m,\gamma) \rangle.$$

Since fans are pairwise disjoint by definition, the naming scheme of the symbols of $S(k,m)$ can be interpreted as assigning to each symbol the index γ of the fan in which it appears, and its index d within that fan.

The construction proceeds by double induction on k and m, as follows.

1. $k = 1$: The sequence is a single fan of size m:

$$S(1,m) = \langle (1,1)\,(2,1) \cdots (m,1) \rangle.$$

Properties (1) and (2) clearly hold here ($C_1(m) = 1$).

2. $k = 2$: The sequence contains a pair of disjoint fans of size m, with a block following each of these fans. Specifically,

$$\begin{aligned} S(2,m) = \langle &(1,1)\,(2,1) \cdots (m-1,1)\,(m,1)\,(m-1,1) \cdots (1,1) \\ &(1,2)\,(2,2) \cdots (m-1,2)\,(m,2)\,(m-1,2) \cdots (1,2) \rangle. \end{aligned}$$

Indeed, $S(2,m)$ contains $C_2(m) = 2$ fans and is composed of $2m$ symbols.

3. $k \ge 3, m = 1$: The sequence is identical to the sequence for $k' = k-1$ and $m' = 2$, except for renaming of its symbols and fans: $S(k-1,2)$ contains $C_{k-1}(2) = \frac{1}{2}C_k(1)$ fans, each of which consists of two symbols; the symbol renaming in $S(k,1)$ causes each of these two elements to become a 1-element fan. Properties (1) and (2) clearly hold.

4. The general case $k \ge 3, m > 1$:

(i) Generate inductively the sequence $S' = S(k,m-1)$; by induction, it contains $\bar{\alpha}$ fans of size $m-1$ each and is composed of $(m-1) \cdot \bar{\alpha}$ symbols.

(ii) Create $\bar{\beta}$ copies of S' whose sets of symbols are pairwise disjoint. For each $\beta \le \bar{\beta}$, rename the symbols in the βth copy S'_β of S' as (d,α,β) where $1 \le d \le m-1$ is the index of the symbol in the fan of S'_β containing it, and $1 \le \alpha \le \bar{\alpha}$ is the index of this fan in S'_β.

(iii) Generate inductively the sequence $S^* = S(k-1,\bar{\alpha})$ whose set of symbols is disjoint from that of any S'_β; by induction, it contains $\bar{\beta}$ fans of size $\bar{\alpha}$ each. Rename the symbols of S^* as (m,α,β) (where α is the index of that symbol within its fan, and β is the index of that fan in S^*). Duplicate the last element $(m,\bar{\alpha},\beta)$ in each of the $\bar{\beta}$ fans of S^*.

(iv) For each $1 \le \alpha \le \bar{\alpha}$, $1 \le \beta \le \bar{\beta}$, extend the αth fan of S'_β by duplicating its last element $(m-1,\alpha,\beta)$, and by inserting the corresponding symbol (m,α,β) of S^* between these duplicated appearances of $(m-1,\alpha,\beta)$. This process extends the $(m-1)$-fans of S'_β into m-fans and adds a new element after each extended fan.

(v) Finally construct the desired sequence $S(k,m)$ by merging the $\bar{\beta}$ copies S'_β of S' with the sequence S^*. This is done by replacing, for each $1 \le \beta \le \bar{\beta}$, the βth fan of S^* by the corresponding copy S'_β of S', as modified in (iv) above. Note that the duplicated copy of the last element in each fan of S^* (formed in step (iii) above) appears now after the copy S'_β that replaces this fan; see Figure 2.1 for an illustration of this process.

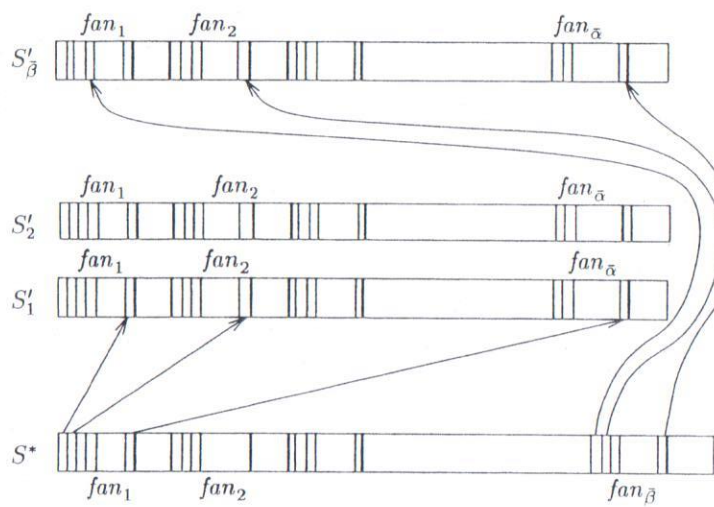

FIGURE 2.1. Merging the subsequences.

To establish property (1), note that $S(k,m)$ consists of

$$\begin{aligned} N_k(m) &= \bar{\beta}\cdot(m-1)\bar{\alpha} + \bar{\alpha}C_{k-1}(\bar{\alpha}) \\ &= (m-1)\bar{\alpha}\bar{\beta} + \bar{\alpha}\bar{\beta} \\ &= m\bar{\alpha}\bar{\beta} \\ &= mC_k(m) \end{aligned}$$

symbols. Property (2) is trivial, because the fans of $S(k,m)$ are precisely the extended fans of the copies S'_β of S', and their number is $C_k(m-1)\cdot\bar{\beta} = C_k(m)$.

We now establish several important properties of the sequences $S(k,m)$. For our present purpose, property (a) is all we need. However, later on in Chapter 4 we will be concerned with geometric realization of the sequences $S(k,m)$, and there we will need to use the other properties.

Theorem 2.20 *For each $k,m \ge 1$ the sequence $S = S(k,m)$ satisfies the following properties:*

(a) *S is a $DS(N_k(m),3)$-sequence.*

(b) *Each symbol of S appears in precisely one fan and makes its first (leftmost) appearance in S.*

(c) *For $k \ge 2$ and for each $\gamma \le \bar{\gamma}$, the last element (m,γ) of the γth fan of S forms the beginning of a contiguous subsequence that is the reverse of that fan:*

$$\langle (m,\gamma)\,(m-1,\gamma)\cdots(2,\gamma)\,(1,\gamma)\rangle.$$

(Note that this sequence is the initial portion of a block of S.)

(d) *For each block c of S, let f be the rightmost fan preceding or including c and let $c_1, c_2, \ldots, c_t$ be the blocks appearing in S between f and c, for some $t \ge 0$. Let a be the first (leftmost) element of c; then either this appearance of a is within f, or else a must also appear in one of the preceding blocks c_i.*

Remark 2.21 (i) For each $\gamma \le \bar{\gamma}$ and each $d < m$, the element (d,γ) in the γth fan of S forms a 1-element block. Note that property (d) is trivially correct for these singleton blocks.

(ii) Property (b) implies in particular that S starts with a fan.

(iii) Unless c is one of the singleton blocks mentioned in (i) above, the first block c_1 in property (d) is the block mentioned in (c) (whose initial portion is the reverse of the fan f). Note that property (d) clearly holds for the case $c = c_1$.

Proof. The proof proceeds by double induction on k and m. The base case $k = 1$ is trivial: $S(1,m)$ is plainly a $DS(m,3)$-sequence, (b) and (d) are trivial, and (c) is vacuous in this case.

The case $k = 2$ is also easy. Here $\bar{\gamma} = 2$ and $S(2,m)$ is obviously a $DS(2m,3)$-sequence, so (a) follows. Properties (b),(c), and (d) are also immediate.

Next consider the case $k > 2$, $m = 1$. Here $S(k,1) = S(k-1,2)$ (with its symbols being renamed), so property (a) holds by induction. Property (b) is also trivial because the only change in the fan structure between $S(k-1,2)$ and $S(k,1)$ is that each fan is split into two subfans. Since each fan is now of size 1, property (c) is trivial too. Finally, since the block structure in $S(k,1)$ is identical to that in $S(k-1,2)$, (d) also follows immediately by induction.

Finally consider the general case $k > 2$, $m > 1$. We first prove property (a). First note that no two adjacent elements of $S = S(k,m)$ are equal: Indeed, by the induction hypothesis, no two adjacent elements either in S^* or in any S'_β are equal; all these sequences have pairwise disjoint sets of symbols, and the