Arrangements in Geometry: Recent Advances and Challenges*

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Abstract. We review recent progress in the study of arrangements in computational and combinatorial geometry, and discuss several open problems and areas for further research.

In this talk I will survey several recent advances in the study of arrangements of curves and surfaces in the plane and in higher dimensions. This is one of the most basic structures in computational and combinatorial geometry. Arrangements appear in a variety of application areas, such as geometric optimization, robotics, graphics and modelling, and molecular biology, just to name a few. Arrangements also possess their own rich structure, which has fueled extensive research for the past 25 years (although, if one wishes, one can find the first trace of them in a study by Steiner in 1826 [37]). While considerable progress has been made, it has left many "hard nuts" that still defy a solution. The aim of this talk is to present these difficult problems, describe what has been done, and what are the future challenges.

An arrangement of a collection S of n surfaces in $\mathbb{R}^d$ is simply the decomposition of d-space obtained by "drawing" the surfaces. More formally, it is the decomposition of d-space into maximal relatively open connected sets, of dimension $0, 1, \ldots, d$, where each set ("face") is contained in the intersection of a fixed subset of the surfaces, and avoids all other surfaces. In many applications, one is interested only in certain substructure of the arrangement, such as lower envelopes, single cells, union of regions, levels, and so on. Other applications study certain constructs related to arrangements, such as incidences between points and curves or surfaces, or cuttings and decompositions of arrangements.

The topics that the talk will aim to address (and, most likely, only partially succeed) include:

(a) Union of geometric objects: In general, the maximum combinatorial complexity of the union of n simply shaped objects in $\mathbb{R}^d$ is $\Theta(n^d)$. However, in many favorable instances better bounds can be established, include unions of fat objects and unions of Minkowski sums of certain kinds; in most cases,

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these bounds are close to $O(n^{d-1})$, which is asymptotically tight. I will briefly review the significant recent progress made on these problems, and list the main challenges that still lie ahead. The main open problems involve unions in three and higher dimensions. For more details, see a recent survey by Agarwal et al. [3].

(b) Decomposition of arrangements: In many algorithmic and combinatorial applications of arrangements, one uses divide-and-conquer techniques, in which the space is decomposed into a small number of regions, each of which is crossed by only a small number of the n given curves or surfaces. Ideally, for a specified parameter r, one seeks a decomposition (also known as a $(1/r)$-*cutting*) into $O(r^d)$ regions, each crossed by at most n/r of the curves or surfaces. This goal has been achieved for planar arrangements, and for arrangements of hyperplanes in any dimesnion. For general simply-shaped surfaces in dimensions three and four, there exist $(1/r)$-cuttings of size close to $O(r^d)$. The problem is wide open in five and higher dimensions. Several (hard) related problems, such as complexity of the overlay of minimization diagrams, or of the sandwich region between two envelopes, will also be mentioned. There is in fact only one method for decomposing arrangements of semi-algebraic surfaces, which is the *vertical decomposition* (see [13] and the many references given below), and the challenge is to understand its maximum combinatorial complexity. For more details, see the book [35], and several surveys on arrangements [5, 6, 34].

(c) Incidences between points and curves and related problems: Bounding the number of incidences between m distinct points and n distinct curves or surfaces has been a major area of research, which traces back to questions raised by Erdős more than 60 years ago [19]. The major milestone in this area is the 1983 paper of Szemerédi and Trotter [39], proving that the maximum number of incidences between m points and n lines in the plane is $\Theta(m^{2/3}n^{2/3} + m + n)$. Since then, significant progress has been made, involving bounds on incidences with other kinds of curves or surfaces, new techniques that have simplified and extended the analysis, and related topics, such as repeated and distinct distances, and other repeated patterns. I will review the state of the art, and mention many open problems. An excellent source of many open problems in this area is the recent monograph of Brass et al. [11]. See also the monographs of Pach and Agarwal [32] and of Matoušek [28], and the survey by Pach and Sharir [33].

(d) k-Sets and levels: What is the maximum possible number of vertices in an arrangement of n lines in the plane, each having exactly k lines passing below it? This simple question is representative of many related problems, for which, in spite of almost 40 years of research, tight answers are still elusive. For example, for the question just asked, the best known upper bound is $O(nk^{1/3})$ [16], and the best known lower bound is $\Omega(n \cdot 2^{c\sqrt{\log k}})$ [30, 41]. Beyond the challenge of tightening these bounds, the same question can be asked for arrangements of hyperplanes in any dimension $d \geq 3$, where the known upper and lower bounds are even wider apart [28, 29, 36], and for arrangements of curves in the plane, where several weaker (but subquadratic) bounds have recetly been established (see, e.g., [12]). I will mention a few of the known results and the implied challenges. Good sources on these problems are Matoušek [28] and a recent survey by Wagner [42].

(e) Generalized Voronoi diagrams: Given a collection S of n *sites* in $\mathbb{R}^d$, and a metric ρ, the Voronoi diagram $Vor_\rho(S)$ is a decomposition of $\mathbb{R}^d$ into cells, one per site, so that the cell of site s consists of all the points for which s is their ρ-nearest neighbor in S. This is one of the most basic constructs in computational geometry, and yet, already in three dimensions, very few sharp bounds are known for the combinatorial complexity of Voronoi diagrams. In three dimensions, the main conjecture is that, under reasonable assumptions concerning the shape of the sites and the metric ρ, the diagram has nearly quadratic complexity. This is a classical result (with tight worst-case quadratic bound) for point sites and the Euclidean metric, but proving nearly quadratic bounds in any more general scenario becomes an extremely hard task, and only very few results are known; see [10, 14, 24, 26]. I will mention the known results and the main challenges. One of my favorites concerns *dynamic Voronoi diagrams* in the plane: If S is a set of n points, each moving at some fixed speed along some line, what is the maximum number of topological changes in the dynamically varying Voronoi diagram of S? The goal is to tighten the gap between the known nearly-cubic upper bound and nearly-quadratic lower bound. See [9, 35] for more details.

(f) Applications to range searching, optimization, and visibility: Arrangements are a fascinating structure to explore for its own sake, but they do have a myriad of applications in diverse areas. As a matter of fact, much of the study of the basic theory of arrangements has been motivated by questions arising in specific applications. I will (attempt to) highlight a few of those applications, and discuss some of the open problems that they still raise.

Bibliography: In addition to the works cited above, the bibliography below is a collection of papers that are relevant to the topics mentioned above. The list is not complete in any sense, but it should give the interested reader sufficiently many pointers into the labyrinth of the literature that has accumulated to date.

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