See "Results formulated", Sect. 1a.

Let $(S_n)_n$ be the simple random walk, and $M_n = \max(S_0, \dots, S_n)$.

**1 Lemma.** The conditional distribution of $S_n$ given $M_n \ge m$ is symmetric around $m$ (for $m \ge 0$).

That is, $\mathbb{E}\big(f(S_n - m)\,\big|\,M_n \ge m\big) = 0$ for every odd function $f$ ('odd' means $f(-x) = -f(x)$).

**2 Exercise.** $\mathbb{P}\big(M_n < m\big) = \mathbb{P}\big(S_n < m\big) - \mathbb{P}\big(S_n > m\big)$ (for $m \ge 0$).

Prove it.

Hint: $f = \operatorname{sgn}$; $\mathbb{E}\,(Y) = \mathbb{E}\big(Y\,\big|\,A\big)\mathbb{P}\big(A\big) + \mathbb{E}\big(Y\,\big|\,\overline{A}\big)\mathbb{P}\big(\overline{A}\big)$.

**3 Exercise.** For $m \ge 0$,

$$\mathbb{P}\big(M_n = m\big) = \mathbb{P}\big(S_n = m\big) + \mathbb{P}\big(S_n = m+1\big) = 2^{-n} \cdot \begin{cases} \binom{n}{\frac{n}{2} \pm \frac{m}{2}} & \text{for } m+n \text{ even,} \\ \binom{n}{\frac{n}{2} \pm \frac{m+1}{2}} & \text{for } m+n \text{ odd.} \end{cases}$$

Prove it.

Hint: $\mathbb{P}\big(M_n < m+1\big) - \mathbb{P}\big(M_n < m\big)$.

**4 Exercise.** $\mathbb{P}\big(S_1 > 0, \dots, S_n > 0\big) = \frac{1}{2}\mathbb{P}\big(S_{n-1} = 0\big) + \frac{1}{2}\mathbb{P}\big(S_{n-1} = 1\big)$.

Prove it.

Hint: $\mathbb{P}\big(S_1 > 0, \dots, S_n > 0\big) = \frac{1}{2}\mathbb{P}\big(M_{n-1} < 1\big)$.

Note that $\mathbb{P}\big(S_{2k} = 0\big) = \mathbb{P}\big(S_1 \ne 0, \dots, S_{2k+1} \ne 0\big) = \mathbb{P}\big(S_1 \ne 0, \dots, S_{2k} \ne 0\big) = \mathbb{P}\big(S_{2k-1} = 1\big)$.

**5 Exercise.** $\mathbb{P}\big(S_n - m = -c, M_n < m\big) = \mathbb{P}\big(S_n - m = -c\big) - \mathbb{P}\big(S_n - m = c\big)$ for $c > 0$, $m \ge 0$.

Prove it.

Hint: $f(c) = 1$, $f(-c) = -1$, otherwise 0; similar to Exercise 2.

In other words, $\mathbb{P}\big(S_n = s, M_n < m\big) = \mathbb{P}\big(S_n = s\big) - \mathbb{P}\big(S_n = 2m - s\big)$. It follows that $\mathbb{P}\big(S_n = s, M_n = m\big) = \mathbb{P}\big(S_n = 2m - s\big) - \mathbb{P}\big(S_n = 2m - s + 2\big)$. The joint distribution is found!

**6 Exercise.** $\mathbb{P}\big(S_1 < 0, \dots, S_n < 0; S_n = -c\big) = \frac{1}{2}\mathbb{P}\big(S_{n-1} = c-1\big) - \frac{1}{2}\mathbb{P}\big(S_{n-1} = c+1\big)$ (for $c \ge 0$).

Prove it.

Hint: it is $\frac{1}{2}\mathbb{P}\big(M_{n-1} < 1, S_{n-1} = -(c-1)\big)$; use Exercise 5.

**7 Exercise.** For $s \ge 0$,

$$\mathbb{P}\big(S_1 > 0, \dots, S_n > 0 \,\big|\, S_n = s\big) = \frac{\mathbb{P}\big(S_{n-1} = s-1\big) - \mathbb{P}\big(S_{n-1} = s+1\big)}{2\mathbb{P}\big(S_n = s\big)} = \frac{s}{n}\,.$$

Prove it.

We get (for $a > b \ge 0$)

$$\mathbb{P}\big(S_1 > 0, \dots, S_{a+b} > 0 \,\big|\, S_{a+b} = a-b\big) = \frac{a-b}{a+b}\,,$$

just the ballot theorem.

Here is another use of reflection.

**8 Exercise.** The expected number of points of increase is equal to 1.

Prove it.

Hint: it is equal to the expected number of points of maxima, defined by

$$\begin{aligned} S_l &< S_k \quad \text{for } l = 0, \dots, k-1\,, \\ S_l &\le S_k \quad \text{for } l = k+1, \dots, n\,. \end{aligned}$$