15 Chart, orientation, volume form

Length of a curve and area of a surface in $\mathbb{R}^3$ are special cases of n-dimensional volume of an n-dimensional manifold in $\mathbb{R}^N$, given infinitesimally by the volume form.

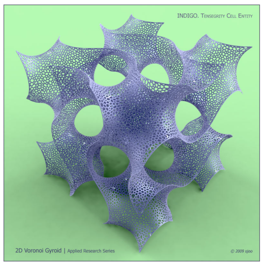


Image: (CC) Jonathan Johanson, `http://cliptic.wordpress.com`

15a Planar curves

Let $M \subset \mathbb{R}^2$ and $(x_0, y_0) \in M$.

Recall that a subset A of M is called a (relative) neighborhood of (x_0, y_0) in M, if A contains all points of M that are close enough to (x_0, y_0). Also, A is (relatively) open in M if it is a neighborhood in M of every point of A.

15a1 Exercise. Assume that G is a neighborhood of 0 in $\mathbb{R}$, $\psi : G \to M$, $\psi(0) = (x_0, y_0)$, ψ is a homeomorphism from G to $\psi(G)$, and $\psi(G)$ is a neighborhood of (x_0, y_0) in M. Prove that $\psi(G_0)$ is a neighborhood of (x_0, y_0) in M for every neighborhood $G_0 \subset G$ of 0 in $\mathbb{R}$.

15a2 Definition. A *chart* (of M around (x_0, y_0)) is a pair (G, ψ) of an open neighborhood G of 0 in $\mathbb{R}$ and a mapping $\psi : G \to M$ such that

(a) $\psi(0) = (x_0, y_0)$;
(b) $\psi(G)$ is an open neighborhood of (x_0, y_0) in M;[1]
(c) ψ is a homeomorphism from G to $\psi(G)$;
(d) $\psi \in C^1(G \to \mathbb{R}^2)$;
(e) $D\psi$ does not vanish (on G).

15a3 Definition. A *co-chart*[2] (of M around (x_0, y_0)) is a pair (U, φ) of an open neighborhood U of (x_0, y_0) in $\mathbb{R}^2$ and a mapping $\varphi : U \to \mathbb{R}$ such that

[1] Relative, of course.
[2] Not a standard terminology.

(a) $\varphi(x_0, y_0) = 0$;[1]
(b) $M \cap U = \{x \in U : \varphi(x) = 0\}$;
(c) $\varphi \in C^1(U)$;
(d) $D\psi$ does not vanish (on U).

In particular, if M is the graph of a function f of class C^1 near x_0, we may take $\psi(t) = \big(x_0+t, f(x_0+t)\big)$ and $\varphi(x, y) = y - f(x)$. The case $x = g(y)$ may be treated similarly. We'll see soon that the general case reduces to these two special cases (locally, but not globally).

15a4 Remark. (a) If (G, ψ) is a chart and $G_0 \subset G$ is an open neighborhood of 0 then $(G_0, \psi|_{G_0})$ is a chart;

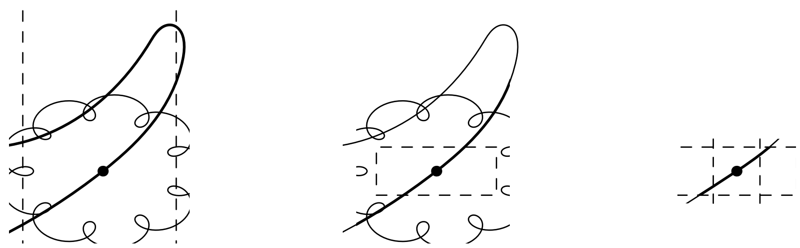
(b) if (U, φ) is a co-chart and $U_0 \subset U$ is an open neighborhood of (x_0, y_0) then $(U_0, \varphi|_{U_0})$ is a co-chart.

15a5 Lemma. Existence of a chart (of M around (x_0, y_0)) is equivalent to existence of a co-chart (of M around (x_0, y_0)).

Proof. "If": given U and φ, we assume that $(D_2\varphi)_{(x_0,y_0)} \ne 0$ (otherwise we swap the coordinates x, y) and apply to φ the implicit function theorem 5c1. Reducing U to some $V \times W$ we get locally a graph

$$M \cap U = \{(x, y) \in V \times W : \varphi(x, y) = 0\} = \{(x, f(x)) : x \in V\}$$

of some function $f : V \to W$ of class C^1. We take $G = V - x_0$, $\psi(x - x_0) = (x, f(x))$ for $x \in G$, and check that (G, ψ) is a chart.



From a chart to a co-chart (and graph).

"Only if": given G and ψ, $\psi(t) = \big(\psi_1(t), \psi_2(t)\big)$, we assume that $\psi_1'(0) \ne 0$ (otherwise we swap the coordinates x, y) and apply to ψ_1 the inverse function theorem 4c1. Reducing G as needed we ensure that ψ_1 is a homeomorphism from G to an open neighborhood V of x_0, and $\psi_1^{-1} : V \to G$ is of class C^1.

[1]This condition may be dropped since it follows from (b).

Taking into account that $\psi(G)$ is a neighborhood of (x_0, y_0) in M, we reduce V and G (again) and choose a neighborhood W of y_0 such that

$$M \cap (V \times W) = \psi(G) \cap (V \times W) \,.$$

We take $U = V \times W$, define $\varphi : U \to R$ by

$$\varphi(x,y) = y - \psi_2\big(\psi_1^{-1}(x)\big) \,,$$

and check that (U, φ) is a co-chart. □

15a6 Definition. A nonempty set $M \subset \mathbb{R}^2$ is a one-dimensional *manifold* (or 1-manifold) if for every $(x_0, y_0) \in M$ there exists a chart of M around (x_0, y_0).

"Co-chart" instead of "chart" gives an equivalent definition due to 15a5.

15a7 Exercise. Which of the following subsets of $\mathbb{R}^2$ are 1-manifolds? Prove your answers, both affirmative and negative.

- $M_1 = \mathbb{R} \times \{0\}$;
- $M_2 = [0,1] \times \{0\}$;
- $M_3 = (0,1) \times \{0\}$;
- $M_4 = \{(0,0)\}$;
- $M_5 = \mathbb{R} \times \{0,1\}$;
- $M_6 = \mathbb{R} \times \mathbb{Z}$;
- $M_7 = \mathbb{R} \times \{1, \frac12, \frac13, \dots\}$;
- $M_8 = M_7 \cup M_1$;
- $M_9 = \{(r\cos\varphi, r\sin\varphi) : 0 < r < 1, \varphi = 1/r\}$;
- $M_{10} = M_9 \cup M_4$;
- $M_{11} = \{(r\cos\varphi, r\sin\varphi) : 0 < r < 1, \varphi = 1/(1-r)\}$;
- $M_{12} = \{(x,y) : x^2 + y^2 = 1\}$;
- $M_{13} = M_{11} \cup M_{12}$.

15b Higher dimensions

Let $M \subset \mathbb{R}^N$, $n \in \{1, \dots, N\}$, and $x_0 \in M$.

15b1 Definition. A *chart* (n-chart of M around x_0) is a pair (G, ψ) of an open neighborhood G of 0 in $\mathbb{R}^n$ and a mapping $\psi : G \to M$ such that

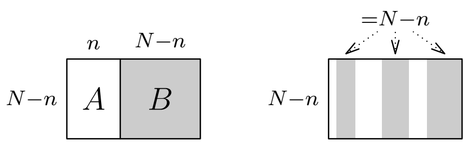
(a) $\psi(0) = x_0$;

(b) $\psi(G)$ is an open neighborhood of x_0 in M;[1]
(c) ψ is a homeomorphism from G to $\psi(G)$;
(d) $\psi \in C^1(G \to \mathbb{R}^N)$;
(e) for every $x \in G$ the linear operator $(D\psi)_x$ from $\mathbb{R}^n$ to $\mathbb{R}^N$ is one-to-one.

15b2 Definition. A *co-chart*[2] (n-cochart of M around x_0) is a pair (U, φ) of an open neighborhood U of x_0 in $\mathbb{R}^N$ and a mapping $\varphi : U \to \mathbb{R}^{N-n}$ such that
(a) $\varphi(x_0) = 0$;[3]
(b) $M \cap U = \{x \in U : \varphi(x) = 0\}$;
(c) $\varphi \in C^1(U \to \mathbb{R}^{N-n})$;
(d) for every $x \in U$ the linear operator $(D\varphi)_x$ from R^N to $\mathbb{R}^{N-n}$ is onto.

In particular, if M is the graph of a mapping $f : \mathbb{R}^n \to \mathbb{R}^{N-n}$ of class C^1 near x_0, that is, $M = \{(u, f(u)) : u \in \mathbb{R}^n\}$, then we may take $\psi(t) = \big(u_0 + t, f(u_0 + t)\big)$ and $\varphi(u, v) = v - f(u)$ for $u \in \mathbb{R}^n$, $v \in \mathbb{R}^{N-n}$; here $(u_0, v_0) = x_0$.

This is one out of $\binom{N}{n}$ similar cases. Recall Sect. 5d: if a linear operator maps $\mathbb{R}^N$ onto $\mathbb{R}^{N-n}$, it does not mean that it is $(\, A \,|\, B \,)$ with invertible B. Some $(N-n) \times (N-n)$ minor is not zero, but not just the rightmost minor. That is, some $N-n$ out of the N variables are functions of the other n variables; but not just the last $N-n$ variables and the first n variables.



15b3 Lemma. Existence of a chart (n-chart of M around x_0) is equivalent to existence of a co-chart (n-cochart of M around x_0).

I skip the proof; it is a straightforward generalization of 15a5.

As before, the general case reduces (locally) to the $\binom{N}{n}$ special cases; some $N-n$ variables are functions of the other n variables. In terms of Sect. 5d, M has a n-chart (or n-cochart) around x_0 if and only if M has n degrees of freedom at x_0.

[1] Relative, of course.
[2] Not a standard terminology.
[3] This condition may be dropped since it follows from (b).

15b4 Exercise. Let (G_1, ψ_1), (G_2, ψ_2) be two n-charts of M around x_0. Prove existence of a mapping $\varphi : G_1 \to G_2$ of class C^1 near 0 such that $\psi_1(u) = \psi_2(\varphi(u))$ for all u near 0, and $\det(D\varphi)_0 \ne 0$.[1]

15b5 Exercise. A relation $\det(D\varphi)_0 > 0$ (for (G_1, ψ_1), (G_2, ψ_2) and φ as above) is an equivalence relation between n-charts of M around x_0. Prove it.

Clearly, there exist exactly two equivalence classes (provided that M has an n-chart around x_0, of course). These equivalence classes are called the two *orientations* of M at x_0.

15b6 Exercise. If M has an n-chart at x_0 then M cannot have an m-chart at x_0 for $m \ne n$. Prove it.[2] However, M can have an m-chart for $m \ne n$ at another point; give an example.

The special status of the point 0 in $\mathbb{R}^n$ is only a matter of convenience; it is easy to reformulate the theory such that $\psi^{-1}(x_0)$ is not necessarily 0.

15b7 Definition. A nonempty set $M \subset \mathbb{R}^N$ is an n-dimensional *manifold* (or n-manifold) if for every $x_0 \in M$ there exists an n-chart of M around x_0.[3]

"Co-chart" instead of "chart" gives an equivalent definition.
A relatively open nonempty subset of an n-manifold is a n-manifold.
An N-manifold in $\mathbb{R}^N$ is just a nonempty open subset of $\mathbb{R}^N$.

15b8 Exercise. (a) If M is an n-manifold in $\mathbb{R}^N$ and $T : \mathbb{R}^N \to \mathbb{R}^N$ an invertible linear operator then $T(M)$ is also an n-manifold; prove it;

(b) for a non-invertible T, $T(M)$ need not be a manifold (of any dimension); give a counterexample.

15b9 Example. [4] Consider the set M of all 3×3 matrices A of the form

$$A = \begin{pmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{pmatrix} \quad \text{for } a, b, c \in \mathbb{R}\,, \ a^2 + b^2 + c^2 = 1\,.$$

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[1] Hint: M has n degrees of freedom at x_0. Values of φ outside a neighborhood of 0 are irrelevant.

[2] Hint: recall 2b13(b).

[3] "In the literature this is usually called a submanifold of Euclidean space. It is possible to define manifolds more abstractly, without reference to a surrounding vector space. However, it turns out that practically all abstract manifolds can be embedded into a vector space of sufficiently high dimension. Hence the abstract notion of a manifold is not substantially more general than the notion of a submanifold of a vector space." Sjamaar, page 69.

[4] The projective plane in disguise.

These are orthogonal projections to one-dimensional subspaces of $\mathbb{R}^3$. We treat M as a subset of the six-dimensional space of all symmetric 3×3 matrices.

The set M is invariant under transformations $A \mapsto UAU^{-1}$ where U runs over all orthogonal matrices (linear isometries); these are linear transformations of the six-dimensional space of matrices. If A corresponds to $x = (a, b, c)$ then UAU^{-1} corresponds to Ux. For arbitrary $A, B \in M$ there exists U such that $UAU^{-1} = B$ ("transitive action").

Thus, M looks the same around all its points ("homogeneous space"). In order to prove that M is a 2-manifold (in $\mathbb{R}^6$) it is sufficient to find a chart (or co-chart) around a single point of M, say,

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M\,.$$

15b10 Exercise. Find a 2-chart of M around A_1.[1]

15b11 Exercise. Locally, near A_1, four coordinates should be smooth functions of the other two coordinates. Which two? Calculate explicitly these four functions of two variables.[2]

Recall the two orientations of M at x_0 introduced after 15b5.

15b12 Definition. (a) An *orientation* of an n-manifold $M \subset \mathbb{R}^N$ is a family $(\mathcal{O}_x)_{x \in M}$ of orientations $\mathcal{O}_x$ of M at points x such that for every $x_0 \in M$ and every $(G, \psi) \in \mathcal{O}_{x_0}$ the relation $(G, \psi) \in \mathcal{O}_x$ holds for all x near x_0.[3]

(b) M is *orientable* if it has (at least one) orientation.

We will see that a sphere is orientable but the Möbius strip is not, as well as M of 15b9. However, a single-chart piece of a manifold is orientable.

An *oriented* manifold is, by definition, a pair $(M, \mathcal{O})$ of a manifold and its orientation. By a chart of an oriented manifold $(M, \mathcal{O})$ we mean a chart (G, ψ) of M such that $(G, \psi) \in \mathcal{O}_x$ for all $x \in \psi(G)$.

15b13 Definition. Let M be an n-manifold in $\mathbb{R}^N$.

(a) A vector $h \in \mathbb{R}^N$ is *tangent* to M at $x_0 \in M$ if $\operatorname{dist}(x_0 + \varepsilon h, M) = o(\varepsilon)$ (as $\varepsilon \to 0$);

(b) the *tangent space* $T_{x_0}M$ (to M at x_0) is the set of all tangent vectors (to M at x_0).

---

[1] Hint: $(b, c) \mapsto (\sqrt{1 - b^2 - c^2}, b, c) = x \mapsto A = \psi(b, c)$.

[2] Hint: solve a quadratic equation.

[3] Of course, $\psi^{-1}(x)$ need not be 0; if this is required, the argument of ψ must be shifted accordingly.

The next exercise shows (in particular) that the tangent space is indeed a vector subspace of $\mathbb{R}^N$.

15b14 Exercise. Let (G,ψ) be a chart around x_0 and (U,φ) a co-chart around x_0. Prove that the following three conditions on a vector $h \in \mathbb{R}^N$ are equivalent:

(a) h is a tangent vector (at x_0);

(b) h belongs to the image of the linear operator $(D\psi)_0 : \mathbb{R}^n \to \mathbb{R}^N$;

(c) h belongs to the kernel of the linear operator $(D\varphi)_{x_0} : \mathbb{R}^N \to \mathbb{R}^{N-n}$.

15b15 Example. Let $M \subset \mathbb{R}^2$ be the graph of a function $f \in C^1(\mathbb{R})$. Then $T_{(x,f(x))}M = \{ \big(\lambda, \lambda f'(x)\big) : \lambda \in \mathbb{R} \}$.

15b16 Exercise. Generalize 15b15 to curves and surfaces in $\mathbb{R}^3$ (that are graphs).

15b17 Definition. A *differential form* of order k (or k-form) on an n-manifold $M \subset \mathbb{R}^N$ is a continuous function ω on the set $\{(x,h_1,\dots,h_k) : x \in M,\, h_1,\dots,h_k \in T_xM\}$ such that for every $x \in M$ the function $\omega(x,\cdot,\dots,\cdot)$ is an antisymmetric multililear k-form on T_xM.

Given a k-form ω on M and a chart (G,ψ) of M, we have the pullback of ω along ψ (similarly to 11f1); this is a k-form $\psi^*\omega$ on G defined by

$$(\psi^*\omega)(u,h_1,\dots,h_k) = \omega\big(\psi(u), (D_{h_1}\psi)_u, \dots, (D_{h_k}\psi)_u\big) \,.$$

In particular, if $k = n$ (the dimension of M) then $\psi^*\omega$ is an n-form on an open set $G \subset \mathbb{R}^n$, therefore

$$\psi^*\omega = f \, du_1 \wedge \dots \wedge du_n$$

for some continuous function $f : G \to \mathbb{R}$. In the spirit of (11f2) we may introduce an improper integral

$$\text{(15b18)} \qquad \int_{(G,\psi)} \omega = \int_G f \,;$$

however, it may diverge.

15c Single-chart integration

15c1 Definition. (a) A k-form ω on an n-manifold $M \subset \mathbb{R}^N$ is *compactly supported* if there exists a compact set $K \subset M$ that supports ω in the sense that $\omega(x,h_1,\dots,h_k) = 0$ for all $x \in M \setminus K$ and $h_1,\dots,h_k \in T_xM$.

(b) ω is a *single-chart form* if there exist a compact set $K \subset M$ that supports ω and a chart (G,ψ) of M such that $K \subset \psi(G)$.

Assume that M, ω, K and (G, ψ) are as in 15c1(b). Then the pullback $\psi^*\omega$ is supported by a compact subset of G. Therefore in the case $k = n$ the integral (15b18) is well-defined as a (proper) Riemann integral (of a compactly supported continuous function on $\mathbb{R}^n$).

The next lemma shows that the formula

(15c2) $$\int_{(M,\mathcal{O})} \omega = \int_{(G,\psi)} \omega$$

is a correct definition of the integral of a single-chart n-form over an oriented n-manifold.

15c3 Lemma. Let ω be a compactly supported n-form on an oriented n-manifold $(M, \mathcal{O})$ in $\mathbb{R}^N$, and (G_1, ψ_1), (G_2, ψ_2) two charts[1] of $(M, \mathcal{O})$ such that $K \subset \psi_1(G_1) \cap \psi_2(G_2)$ for some compact K that supports ω. Then

$$\int_{(G_1,\psi_1)} \omega = \int_{(G_2,\psi_2)} \omega \,.$$

Proof. The set $\tilde G = \psi_1(G_1) \cap \psi_2(G_2)$ is (relatively) open in M, therefore sets $\tilde G_1 = \psi_1^{-1}(\tilde G) \subset G_1$, $\tilde G_2 = \psi_2^{-1}(\tilde G) \subset G_2$ are open (in $\mathbb{R}^n$). A mapping $\varphi : \tilde G_1 \to \tilde G_2$, $\varphi(u) = \psi_2^{-1}(\psi_1(u))$ is a diffeomorphism by 15b4. The equality

$$\psi_1 = \psi_2 \circ \varphi \quad \text{on } \tilde G_1$$

implies

$$\psi_1^*\omega = \varphi^*(\psi_2^*\omega) \quad \text{on } \tilde G_1$$

by the chain rule.[2] We have $\psi_1^*\omega = f_1 \, du_1 \wedge \dots \wedge du_n$, $\psi_2^*\omega = f_2 \, du_1 \wedge \dots \wedge du_n$ for some $f_1 \in C(\tilde G_1)$, $f_2 \in C(\tilde G_2)$. Thus,

$$f_1 \, du_1 \wedge \dots \wedge du_n = \varphi^*\big(f_2 \, du_1 \wedge \dots \wedge du_n\big) = (f_2 \circ \varphi) \, d\varphi_1 \wedge \dots \wedge d\varphi_n$$

where $\varphi_i = u_i \circ \varphi$. It follows that $f_1(u) = f_2(\varphi(u)) \det(D\varphi)_u$ for all $u \in \tilde G_1$. Using Theorem 8a5, $\int_{G_2} f_2 = \int_{\tilde G_2} f_2 = \int_{\tilde G_1} (f_2 \circ \varphi) |\det D\varphi| = \int_{\tilde G_1} (f_2 \circ \varphi) \det D\varphi = \int_{\tilde G_1} f_1 = \int_{G_1} f_1$. □

15d Volume form

All antisymmetric multililear n-forms L on $\mathbb{R}^n$ are the same up to a coefficient,

$$L = c \, dx_1 \wedge \dots \wedge dx_n \quad \text{for some } c \in \mathbb{R} \,;$$
$$L(a_1, \dots, a_n) = c \det(a_1, \dots, a_n) \quad \text{for all } a_1, \dots, a_n \in \mathbb{R}^n \,.$$

[1] Orientation must be respected.

[2] This is similar to the equality $(\varphi \circ \Gamma)^*\omega = \Gamma^*(\varphi^*\omega)$ in Sect. 11f.

If $a_1, \dots, a_n$ are an orthonormal basis then $\det(a_1, \dots, a_n) = \pm 1$, and therefore $|L(a_1, \dots, a_n)| = |c|$ does not depend on the basis.

Thus, for every n-dimensional vector space V, all antisymmetric multililear n-forms on V are a one-dimensional vector space, — a line. The two rays of this line are, by definition, the two orientations of V. In other words, the two orientations of V are the two equivalence classes of nontrivial (that is, not identically zero) antisymmetric multililear n-forms on V; the equivalence relation is, $\exists c > 0 \; L_1 = cL_2$.

For an n-dimensional Euclidean space E, each orientation contains exactly one L *normalized* in the sense that $|L(a_1, \dots, a_n)| = 1$ for some (therefore, every) orthonormal basis $a_1, \dots, a_n$ of E.

If $M \subset \mathbb{R}^N$ is an n-manifold and $x_0 \in M$, then the two orientations of M at x_0 correspond to the two orientations of $T_{x_0}M$; namely, an n-chart (G, ψ) of M at x_0 corresponds to an antisymmetric multililear n-form L on $T_{x_0}M$ if $L\big((D_1\psi)_0, \dots, (D_n\psi)_0\big) > 0$.

15d1 Definition. An n-form μ on an oriented n-manifold $(M, \mathcal{O})$ in $\mathbb{R}^N$ is the *volume form,* if for every $x \in M$ the antisymmetric multililear n-form $\mu(x, \cdot, \dots, \cdot)$ is normalized and corresponds to the orientation $\mathcal{O}_x$.

Clearly, such μ is unique. Is it clear that μ exists? Surely, $\mu(x, \cdot, \dots, \cdot)$ is well-defined for each x; but is it continuous in x? We will arrive soon to a useful explicit formula for μ in terms of a chart, thus getting existence as a byproduct. For now, taking existence for granted, we use μ in the following definition.

15d2 Definition. The integral of a single-chart continuous function $f : M \to \mathbb{R}$ over an oriented manifold $(M, \mathcal{O})$ is

$$\int_{(M,\mathcal{O})} f = \int_{(M,\mathcal{O})} f\mu$$

where μ is the volume form on $(M, \mathcal{O})$.

15d3 Example. Let $M \subset \mathbb{R}^2$ be the graph of a function $f \in C^1(\mathbb{R})$. The whole M is covered by a chart $\mathbb{R} = G_+ \ni x \mapsto \psi_+(x) = (x, f(x)) \in M$; denote by $\mathcal{O}_+$ the corresponding orientation of M, and by $\mathcal{O}_-$ the other orientation. The two volume forms on M are $\mu_\pm\big((x, f(x)), (\lambda, \lambda f'(x))\big) = \pm\lambda\sqrt{1 + f'^2(x)}$; thus, $\psi_+^*\mu_+ = \sqrt{1 + f'^2}\,dx$. Given a compactly supported function $g \in C(M)$, we have

$$\int_{(M,\mathcal{O}_+)} g = \int_{\mathbb{R}} g(x, f(x))\sqrt{1 + f'^2(x)}\,\mathrm{d}x\,.$$

Another chart $\mathbb{R} = G_- \ni x \mapsto \psi_-(x) = (-x, f(-x)) \in M$ corresponds to $\mathcal{O}_-$; we have $\psi_-^* \mu_- = \sqrt{1 + f'(-x)^2}\, dx$ (think, why not "$-\sqrt{\dots}$"); thus,

$$\int_{(M,\mathcal{O}_-)} g = \int_{\mathbb{R}} g(-x, f(-x)) \sqrt{1 + f'^2(-x)}\, \mathrm{d}x \,,$$

the same result for the other orientation.

Can we generalize 15d3 to a surface M in $\mathbb{R}^3$ (the graph of a function $f \in C^1(\mathbb{R}^2)$)? We know the tangent space (recall 15b16) $T_{(x,y,f(x,y))}M$, it is spanned by two vectors, $(1, 0, (D_1 f)_{(x,y)})$ and $(0, 1, (D_2 f)_{(x,y)})$, but they are not orthogonal. We may apply the orthogonalization process, but it leads to unpleasant formulas even for $n = 2$ (and the more so for higher n). Fortunately a better way exists.

For arbitrary n vectors $a_1, \dots, a_n \in \mathbb{R}^n$,

$$\big(\det(a_1, \dots, a_n)\big)^2 = \big(\det(A)\big)^2 = \det(A^{\mathrm{t}} A) = \\ = \det\big(\langle a_i, a_j \rangle\big)_{i,j} = \begin{vmatrix} \langle a_1, a_1 \rangle & \dots & \langle a_1, a_n \rangle \\ \langle a_2, a_1 \rangle & \dots & \langle a_2, a_n \rangle \\ \dots & \dots & \dots \\ \langle a_n, a_1 \rangle & \dots & \langle a_n, a_n \rangle \end{vmatrix} \,;$$

here $A = (\, a_1 \,|\, \dots \,|\, a_n \,)$ is the matrix whose columns are the vectors $a_1, \dots, a_n$; accordingly, $A^{\mathrm{t}} A$ is the matrix of scalar products (think, why), the so-called Gram matrix, and its determinant is called the Gram determinant, or Gramian of $a_1, \dots, a_n$.

Let $E \subset \mathbb{R}^N$ be an n-dimensional subspace, $e_1, \dots, e_n$ its orthonormal basis, and L a normalized antisymmetric multililear n-form on E. How to calculate $|L(h_1, \dots, h_n)|$ for arbitrary $h_1, \dots, h_n \in E$? By the Gramian:

$$|L(h_1, \dots, h_n)| = \sqrt{\det\big(\langle h_i, h_j \rangle\big)_{i,j}} \,. \tag{15d4}$$

Here is why. Consider a linear isometry $T : \mathbb{R}^n \to E$, $T(u_1, \dots, u_n) = u_1 e_1 + \dots + u_n e_n$. The antisymmetric multililear n-form $(a_1, \dots, a_n) \mapsto L(Ta_1, \dots, Ta_n)$ on $\mathbb{R}^n$ returns $L(e_1, \dots, e_n) = \pm 1$ on the usual basis of $\mathbb{R}^n$; therefore

$$L(Ta_1, \dots, Ta_n) = \pm \det(a_1, \dots, a_n) \quad \text{for all } a_1, \dots, a_n \in \mathbb{R}^n \,.$$

Taking $a_1, \dots, a_n$ such that $Ta_1 = h_1, \dots, Ta_n = h_n$ we get

$$\big(L(h_1, \dots, h_n)\big)^2 = \big(\det(a_1, \dots, a_n)\big)^2 = \det\big(\langle a_i, a_j \rangle\big)_{i,j} = \det\big(\langle h_i, h_j \rangle\big)_{i,j} \,,$$

since T is isometric.

Thus, in order to check whether an antisymmetric multililear n-form L on an n-dimensional $E \subset \mathbb{R}^N$ is normalized or not, we do not need an orthonormal basis in E. It suffices to have linearly independent vectors $h_1, \dots, h_n \in E$ and check (15d4).

If μ is a volume form on $(M, \mathcal{O})$ and (G, ψ) a chart of $(M, \mathcal{O})$ then the pullback $\psi^* \mu$ satisfies

$$(\psi^*\mu)(u, e_1, \dots, e_n) = \mu\big(\psi(u), (D_1\psi)_u, \dots, (D_1\psi)_u\big) = J_\psi(u) \,,$$

where $e_1, \dots, e_n$ are the usual basis of $\mathbb{R}^n$, and

$$J_\psi(u) = \sqrt{\det\big(\langle (D_i\psi)_u, (D_j\psi)_u \rangle\big)_{i,j}}$$

is the (generalized) Jacobian of ψ. We see that

(15d5) $$\psi^*\mu = J_\psi \, du_1 \wedge \dots \wedge du_n \,.$$

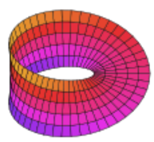
Now, given $(M, \mathcal{O})$ and (G, ψ) (but not μ), we can construct a form μ on the oriented n-manifold $\psi(G) \subset M$ satisfying (15d5), namely, $\mu = (\psi^{-1})^*(J_\psi \, du_1 \wedge \dots \wedge du_n)$; existence of the volume form is thus proved (on every orientable manifold, not just single-chart). We have

(15d6) $$\int_{(M,\mathcal{O})} f = \int_{(M,\mathcal{O})} f\mu = \int_{(G,\psi)} f\mu = \int_G (f \circ \psi) J_\psi$$

for every continuous $f : M \to \mathbb{R}$ supported by a compact $K \subset \psi(G)$.

15d7 Exercise. Consider a Möbius strip (without the edge),

$$M = \{\Gamma(s, \theta) : s \in (-1, 1), \, \theta \in [0, 2\pi]\} \,,$$
$$\Gamma(s, \theta) = \begin{pmatrix} (R + rs\cos\frac{\theta}{2})\cos\theta \\ (R + rs\cos\frac{\theta}{2})\sin\theta \\ rs\sin\frac{\theta}{2} \end{pmatrix} ,$$



for given $R > r > 0$ (as in Sect. 12b). Prove that it is a non-orientable 2-manifold in $\mathbb{R}^3$.[1]

Two facts without proofs: every 1-manifold in $\mathbb{R}^N$ is orientable; every *compact* 2-manifold in $\mathbb{R}^3$ is orientable.

15d8 Exercise. Continuing 15b9 prove that the compact 2-manifold $M \subset \mathbb{R}^6$ is non-orientable.[2]

[1] Hint: think about the function $\theta \mapsto \mu\big(\Gamma(0, \theta), D_1\Gamma(0, \theta), D_2\Gamma(0, \theta)\big)$.

[2] Hint: similar to 15d7. (In fact, a part of M is diffeomorphic to the Möbius strip.)

15d9 Exercise. Let $f \in C^1(\mathbb{R})$, M_a be the graph of $f(\cdot)+a$ for $a \in \mathbb{R}$, and $g \in C(\mathbb{R}^2)$ compactly supported. Prove that

(a) $\int_{\mathbb{R}} \mathrm{d}a \int_{M_a} g^2 \ge \int_{\mathbb{R}^2} g^2$;

(b) the equality holds if and only if $\forall x, y \;\; f'(x)g(x,y) = 0$.

15d10 Exercise. Find J_ψ given $\psi(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Compare your answer with (14b11).

15d11 Exercise. Find J_ψ given $\psi(x) = \big(x, \sqrt{1-|x|^2}\big) \in \mathbb{R}^{n+1}$ for $x \in \mathbb{R}^n$, $|x| < 1$.

Answer: $1/\sqrt{1-|x|^2}$.

15d12 Exercise. Consider a half-space $G = \mathbb{R}^{n-1} \times (0,\infty) \subset \mathbb{R}^n$, semispheres $M_r = \{x \in G : |x| = r\}$ for $r > 0$, and a compactly supported $f \in C(G)$. Prove that

(a) $\displaystyle \int_{M_r} f = \int_{\{u\in\mathbb{R}^{n-1}:|u|<r\}} \frac{r}{\sqrt{r^2-|u|^2}} f\big(u, \sqrt{r^2-|u|^2}\big)\,\mathrm{d}u$;

(b) $\displaystyle \int_0^\infty \mathrm{d}r \int_{M_r} f = \int_G f$.

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