Solutions to selected exam questions

Question 3.
A vector field $F \in C^1(\mathbb{R}^3 \to \mathbb{R}^3)$ satisfies

$$\operatorname{div} F = 0\,;$$
$$F(x,y,z) \ne 0 \quad \Longrightarrow \quad 0 \le (x^2+y^2)^{3/2} z \le 1\,,$$
$$F(x,y,z) = o(x^2+y^2+z^2)\,, \quad x^2+y^2+z^2 \to \infty\,.$$

Consider the flux of F through the plane $z = c$. Prove that this flux equals zero for every $c \in \mathbb{R}$.

Solution.
First, F vanishes outside the set $\{(x,y,z) : 0 \le (x^2+y^2)^{3/2} z \le 1\}$ and therefore, by continuity, F vanishes also on the boundary of this set. That is, F vanishes outside the open set $\{(x,y,z) : z > 0, (x^2+y^2)^{3/2} z < 1\}$. The case $c \le 0$ is thus trivial. Now let $c > 0$.

Here is the idea. For arbitrary $R > 0$ we consider the cylinder

$$\{(x,y,z) : x^2+y^2 \le R^2, 0 \le z \le c\}\,.$$

The flux of F through its surface is zero, since $\operatorname{div} F = 0$. The surface of the cylinder consists of two disks

$$\{(x,y,0) : x^2+y^2 \le R^2\}\,, \quad \{(x,y,c) : x^2+y^2 \le R^2\}\,,$$

and the lateral surface

$$\{(x,y,z) : x^2+y^2 = R^2, 0 \le z \le c\}\,.$$

The flux through the top disk is the flux through the plane $z = c$ when R is large enough (namely, $R \ge c^{-1/3}$. The flux through the bottom disk is 0 (since $F = 0$ here). Thus, it is sufficient to prove that the flux through the lateral surface converges to 0 as $R \to \infty$.

On the lateral surface, first, $F = o(R^2)$, and second, $F \ne 0$ only when $z < 1/R^3$. The relevant area is $2\pi R \cdot \frac{1}{R^3} = 2\pi/R^2$. Thus, the flux is $o(R^2 \cdot 2\pi/R^2) = o(1)$.

More formally, we may treat the cylinder as a singular 3-box $\Gamma : [0,R] \times [0,2\pi] \times [0,c] \to \mathbb{R}^3$,

$$\Gamma(r,\theta,z) = (r\cos\theta, r\sin\theta, z)\,.$$

Alternatively, instead of the cylinder we may use the (non-singular) box $[-R,R] \times [-R,R] \times [0,c]$.