15 From boundary to exterior derivative

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Terms "boundary" and "derivative" get new meaning, and become dual to each other.

15a Chains

Recall the integral $\int_\Gamma \omega$ defined by (11e12).

15a1 Definition. A (singular) k-chain (in $\mathbb{R}^n$) is a formal linear combination of singular k-boxes.

That is,

$$C = c_1\Gamma_1 + \dots + c_p\Gamma_p \,,$$

where $c_1,\dots,c_p \in \mathbb{R}$, and $\Gamma_1,\dots,\Gamma_p$ are singular k-boxes. More formally, this is a real-valued function with finite support on the (huge!) set of all singular k-boxes;

$$c_1 = C(\Gamma_1),\dots,c_p = C(\Gamma_p)\,; \quad C(\Gamma) = 0 \text{ for all other } \Gamma\,.$$

Clearly, all k-chains are a (huge) vector space, with a basis indexed by all singular k-boxes. Less formally we say that the singular k-boxes *are* the basis, and each singular box is (a special case of) a chain: $\Gamma = 1\cdot\Gamma$.

15a2 Definition.

$$\int_C \omega = c_1 \int_{\Gamma_1} \omega + \dots + c_p \int_{\Gamma_p} \omega$$

for every k-chain $C = c_1\Gamma_1 + \dots + c_p\Gamma_p$ and every k-form ω.

Note that the integral is bilinear; $\int_C \omega$ is linear in C for every ω (by construction), and linear in ω for every C (since $\int_\Gamma \omega$ evidently is linear in ω).

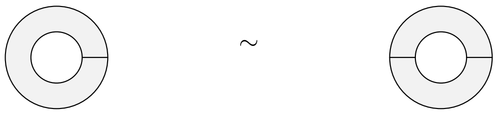
15a3 Definition. Two k-chains C_1, C_2 are *equivalent* if

$$\int_{C_1} \omega = \int_{C_2} \omega \quad \text{for all } k\text{-forms } \omega \text{ (of class } C^0) \,.$$

Let $B \subset \mathbb{R}^k$ be a box, P its partition, and $\Gamma : B \to \mathbb{R}^n$ a singular box. Then

$$\Gamma \sim \sum_{b \in P} \Gamma|_b \,,$$

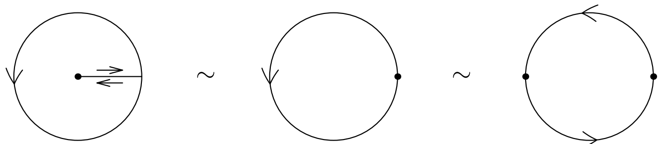
since $\Gamma \mapsto \int_\Gamma \omega$ is an additive function of a singular box.



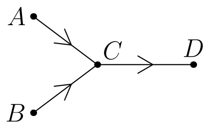
Recall that singular 1-boxes are C^1-paths.

By 11c13, equivalent paths are equivalent 1-chains.

By 11c11, the 1-chain $\gamma + \gamma_{-1}$ is equivalent to 0; here γ_{-1} is the inverse path.



(recall 11e2)



$ACD + BC \sim AC + BCD \sim AC + BC + CD$

15b Order 0 and order 1

The case $k = 0$ is included as follows. The space $\mathbb{R}^0$ consists, by definition, of a single point 0. The only 0-dimensional box is $\{0\}$. A singular 0-box in $\mathbb{R}^n$ is thus $\{x\}$ for some $x \in \mathbb{R}^n$.[1] A 0-form on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \to \mathbb{R}$ (of class C^m). And

$$\int_{\{x\}} \omega = \omega(x) \,,$$

[1]Well, more formally, it is $\{(0, x)\}$.

of course. Accordingly, $\int_C \omega = c_1\omega(x_1) + \dots + c_p\omega(x_p)$ for a 0-chain $C = c_1\{x_1\} + \dots + c_p\{x_p\}$.

15b1 Exercise. If two 0-chains are equivalent then they are equal.
Prove it.

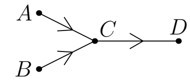
The *boundary* of a singular 1-box $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is, by definition, the 0-chain

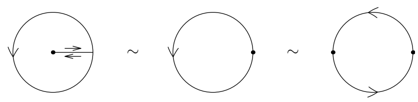
$$\partial\gamma = \{\gamma(t_1)\} - \{\gamma(t_0)\} \,,$$

a linear combination of two singular 0-boxes (not to be confused with $\gamma(t_1) - \gamma(t_0)$). Thus,

$$\int_{\partial\gamma} \omega = \omega(\gamma(t_1)) - \omega(\gamma(t_0)) \quad \text{for a 0-form } \omega \,.$$

The boundary of a 1-chain $C = c_1\gamma_1 + \dots + c_p\gamma_p$ is, by definition, the 0-chain $\partial C = c_1\partial\gamma_1 + \dots + c_p\partial\gamma_p$. For example,

the boundary of  is $-\{A\} - \{B\} + \{C\} + \{D\}$;

the boundary of  is 0.

Note that the map $C \mapsto \partial C$ is linear (by construction).

Given a 0-form ω of class C^1 on $\mathbb{R}^n$, that is, a continuously differentiable function $\omega : \mathbb{R}^n \to \mathbb{R}$, its derivative $D\omega$ may be thought of as a 1-form of class C^0 on $\mathbb{R}^n$, denoted $d\omega$;

$$(d\omega)(x,h) = (D\omega)_x(h) = (D_h\omega)_x \,. \tag{15b2}$$

15b3 Proposition. (*Stokes' theorem for $k = 1$*)
Let C be a 1-chain in $\mathbb{R}^n$, and ω a 0-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C} \omega \,.$$

Proof. By linearity in C it is sufficient to prove it for $C = \gamma$ (a single 1-box, that is, a path $\gamma : [t_0, t_1] \to \mathbb{R}^n$). We have

$$\int_\gamma d\omega = \int_{t_0}^{t_1} d\omega\bigl(\gamma(t), \gamma'(t)\bigr)\,\mathrm{d}t = \int_{t_0}^{t_1} (D\omega)_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t =$$
$$= \int_{t_0}^{t_1} \Bigl(\frac{\mathrm{d}}{\mathrm{d}t}\omega(\gamma(t))\Bigr)\,\mathrm{d}t = \omega(\gamma(t_1)) - \omega(\gamma(t_0)) = \int_{\partial\gamma} \omega \,.$$

$\square$

15b4 Corollary.

$$C_1 \sim C_2 \quad \text{implies} \quad \partial C_1 = \partial C_2$$

for arbitrary 1-chains C_1, C_2 in $\mathbb{R}^n$.

Indeed, $\int_{\partial C_1} \omega = \int_{C_1} d\omega = \int_{C_2} d\omega = \int_{\partial C_2} \omega$ for every 0-form ω of class C^1. Similarly to 15b1 it follows that $\partial C_1 = \partial C_2$.

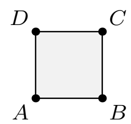
The case $k = 1$ is special; for higher k we'll see (in 16e9) that $C_1 \sim C_2$ implies $\partial C_1 \sim \partial C_2$ but not $\partial C_1 = \partial C_2$. Nothing like 15b1 holds for higher k.

It is easy to prove that $C_1 \sim C_2 \implies \partial C_1 \sim \partial C_2$ for $k = 1$ without 15b1. The only problem is that $C^1(\mathbb{R}^n) \ne C^0(\mathbb{R}^n)$. However, $C^1(\mathbb{R}^n)$ is dense in $C^0(\mathbb{R}^n)$ (recall 7d28).

15c Order 1 and order 2

The boundary of a singular 2-box Γ is, by definition, the 1-chain

$$\Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA} = \Gamma|_{AB} + \Gamma|_{BC} - \Gamma|_{DC} - \Gamma|_{AD} \,.$$



This is not really a definition of a 1-chain, since I did not specify the four 1-dimensional boxes (which is very easy to do); but its equivalence class is well-defined, and this is all we need for the following question.

Given a 1-form ω, can we construct a 2-form, call it $d\omega$, such that $\int_C d\omega = \int_{\partial C} \omega$ for all 2-chains C?

We have a function $\Gamma \mapsto \int_{\partial\Gamma} \omega$ of a singular box; this is an additive function, since the map $\Gamma \mapsto \partial\Gamma$ is additive (up to equivalence).

We want to differentiate this additive function in the hope that its derivative exists and is a 2-form $d\omega$.

Note that

(15c1) $$\partial(\partial\Gamma) \sim 0 \quad \text{for a singular 2-box } \Gamma$$

(try it for Γ of 11e2 and 11e3). By 15b3, $\int_{\partial\Gamma} d\omega = \int_{\partial(\partial\Gamma)} \omega = 0$ for every 0-form ω of class C^1. It should be $\int_\Gamma d(d\omega) = \int_{\partial\Gamma} d\omega = 0$ for all Γ, that is,

$d(d\omega) = 0$ for every 0-form ω of class C^2.[1] A wonder: the second derivative of a 0-form is always zero, irrespective of the second derivatives of the function! Indeed, exterior derivative is very similar to the usual derivative for 0-forms, but very dissimilar for 1-forms.

Existence of $d\omega$ is the point of Stokes' theorem 15c3. For now we'll find a necessary condition on $d\omega$ that ensures its uniqueness and provides an explicit formula.

Given a point $x \in \mathbb{R}^n$ and two vectors $h, k \in \mathbb{R}^n$, we consider small singular boxes $\Gamma_\varepsilon : [0,1] \times [0,1] \to \mathbb{R}^n$,

$$\Gamma_\varepsilon(u_1, u_2) = x + \varepsilon u_1 h + \varepsilon u_2 k\,;$$

an additive function on Γ_ε should be of order ε^2 as $\varepsilon \to 0+$; we divide it by ε^2 and calculate the limit:

$$\begin{gathered}\frac{1}{\varepsilon^2}\int_{\partial\Gamma_\varepsilon}\omega = \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon u_1 h, \varepsilon h)\,\mathrm{d}u_1 + \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon h + \varepsilon u_2 k, \varepsilon k)\,\mathrm{d}u_2 - \\ - \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon u_1 h + \varepsilon k, \varepsilon h)\,\mathrm{d}u_1 - \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon u_2 k, \varepsilon k)\,\mathrm{d}u_2 = \\ = \int_0^1 \frac{\omega(x+\varepsilon u_1 h, h) - \omega(x+\varepsilon u_1 h + \varepsilon k, h)}{\varepsilon}\,\mathrm{d}u_1 + \\ + \int_0^1 \frac{\omega(x+\varepsilon h + \varepsilon u_2 k, k) - \omega(x+\varepsilon u_2 k, k)}{\varepsilon}\,\mathrm{d}u_2 \to -\big(D_k\omega(\cdot,h)\big)_x + \big(D_h\omega(\cdot,k)\big)_x\,,\end{gathered}$$

assuming $\omega \in C^1$. Taking into account that

$$\frac{1}{\varepsilon^2}\int_{\Gamma_\varepsilon} d\omega \to (d\omega)(x,h,k)$$

(for arbitrary 2-form in place of $d\omega$) we see that the needed $d\omega$ (if exists) is as follows.

15c2 Definition. The *exterior derivative* of a 1-form ω of class C^1 is the 2-form $d\omega$ defined by

$$(d\omega)(\cdot,h,k) = D_h\omega(\cdot,k) - D_k\omega(\cdot,h)\,.$$

15c3 Theorem. (*Stokes' theorem for $k=2$*)

Let C be a 2-chain in $\mathbb{R}^n$, and ω a 1-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C}\omega\,.$$

[1] This fact will be proved for all forms of all orders, see 16e4(b).

This is a special case of Theorem 15f3, to be proved much later.

15c4 Exercise. For a 1-form $\omega = f(x,y)\,dx + g(x,y)\,dy$ on $\mathbb{R}^2$ (or an open subset of $\mathbb{R}^2$) prove that $(d\omega)(\cdot,h,k) = (D_1 g - D_2 f)\det(h,k)$, that is, $d\omega = (D_1 g - D_2 f)\mu_2$, where μ_2 is the volume form on $\mathbb{R}^2$.

15c5 Exercise. For the form $\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$ (treated in Sect. 11d) on $\mathbb{R}^2 \setminus \{0\}$ prove that $d\omega = 0$, but $\int_\gamma \omega \ne 0$ for some γ; does it contradict 15c3?

15c6 Exercise. For the form $\omega = \frac{-y\,dx + x\,dy}{2}$ (mentioned in Sect. 11d) on $\mathbb{R}^2$ prove that $d\omega = \mu_2$. Reconsider 11d2 in the light of 15c3.

15d Order $N-1$: forms and vector fields

Recall two types of integral over an n-manifold:

* of an n-form ω, $\int_{(M,\mathcal{O})} \omega$, defined by (12c2)-(13a4);
* of a function f, $\int_M f$, defined by (13a7)-(13a8);

they are related by

$$\int_M f = \int_{(M,\mathcal{O})} f\mu_{(M,\mathcal{O})} \,,$$

where $\mu_{(M,\mathcal{O})}$ is the volume form; that is, $\int_M f = \int_{(M,\mathcal{O})} \omega$ where $\omega = f\mu_{(M,\mathcal{O})}$. Interestingly, every n-form ω on an orientable n-manifold $M \subset \mathbb{R}^N$ is $f\mu_{(M,\mathcal{O})}$ for some $f \in C(M)$. This is a consequence of the one-dimensionality[1] of the space of all antisymmetric multilinear n-forms on the tangent space $T_x M$. We have $f(x) = \omega(x, e_1, \dots, e_n)$ for some (therefore, every) orthonormal basis $(e_1, \dots, e_n)$ of $T_x M$ that conforms to $\mathcal{O}_x$. But if ω is defined on the whole $\mathbb{R}^N$ (not just on M), it does not lead to a function f on the whole $\mathbb{R}^N$; indeed, in order to find $f(x)$ we need not just x but also $T_x M$ (and its orientation).

The case $n = N$ is simple: every N-form ω on $\mathbb{R}^N$ (or on an open subset of $\mathbb{R}^N$) is $f\mu_N$ (for some continuous f), where μ_N is the volume form on $\mathbb{R}^N$; that is,

$$\begin{aligned} \mu_N(x, h_1, \dots, h_N) &= \det(h_1, \dots, h_N)\,; \\ \omega(x, h_1, \dots, h_N) &= f(x)\det(h_1, \dots, h_N)\,; \\ f(x) &= \omega(x, e_1, \dots, e_N)\,. \end{aligned}$$

We turn to the case $n = N - 1$.

[1] Recall Sect. 11e and 12c.

The space of all antisymmetric multilinear n-forms L on $\mathbb{R}^N$ is of dimension $\binom{N}{n} = N$. Here is a useful linear one-to-one correspondence between such L and vectors $h \in \mathbb{R}^N$:

$$\forall h_1, \dots, h_n \;\; L(h_1, \dots, h_n) = \det(h, h_1, \dots, h_n)\,.$$

Introducing the cross-product $h_1 \times \dots \times h_n$ by[1]

$$\forall h \;\; \langle h, h_1 \times \dots \times h_n \rangle = \det(h, h_1, \dots, h_n)$$

(it is a vector orthogonal to $h_1, \dots, h_n$), we get

$$L(h_1, \dots, h_n) = \langle h, h_1 \times \dots \times h_n \rangle\,.$$

Doing so at every point, we get a linear one-to-one correspondence between n-forms ω on $\mathbb{R}^N$ and vector fields F on $\mathbb{R}^N$:

$$\omega(x, h_1, \dots, h_n) = \langle F(x), h_1 \times \dots \times h_n \rangle\,. \tag{15d1}$$

Similarly, $(n-1)$-forms ω on an oriented n-dimensional manifold $(M, \mathcal{O})$ in $\mathbb{R}^N$ (not just $N - n = 1$) are in a linear one-to-one correspondence with *tangent* vector fields F on M, that is, $F \in C(M \to \mathbb{R}^N)$ such that $\forall x \in M \;\; F(x) \in T_x M$.

Let $M \subset \mathbb{R}^N$ be an orientable n-manifold (still, $n = N - 1$), ω and F as in (15d1). We know that $\omega|_M = f \mu_{(M,\mathcal{O})}$ for some f. How is f related to F? Given $x \in M$, we take an orthonormal basis $(e_1, \dots, e_n)$ of $T_x M$, note that $e_1 \times \dots \times e_n = \mathbf{n}_x$ is a unit normal vector to M at x, and

$$\begin{aligned}\langle F(x), \mathbf{n}_x \rangle = \langle F(x), e_1 \times \dots \times e_n \rangle = \omega(x, e_1, \dots, e_n) = \\ = f(x) \mu_{(M,\mathcal{O})}(x, e_1, \dots, e_n) = \pm f(x)\,.\end{aligned}$$

In order to get "+" rather than "±" we need a coordination between the orientation $\mathcal{O}$ and the normal vector $\mathbf{n}_x$. Let the basis $(e_1, \dots, e_n)$ of $T_x M$ conform to the orientation $\mathcal{O}_x$ (of M at x, or equivalently, of $T_x M$, recall Sect. 12b), then $\mu_{(M,\mathcal{O})}(x, e_1, \dots, e_n) = +1$. The two unit normal vectors being $\pm e_1 \times \dots \times e_n$, we say that $\mathbf{n}_x = e_1 \times \dots \times e_n$ conforms to the given orientation, and get[2]

$$\langle F(x), \mathbf{n}_x \rangle = f(x)\,; \quad \omega|_M = \langle F, \mathbf{n} \rangle \mu_{(M,\mathcal{O})}\,.$$

[1] For $N = 3$ the cross-product is a binary operation, but for $N > 3$ it is not. In fact, it is possible to define the corresponding associative binary operation (the so-called exterior product, or wedge product), not on vectors but on the so-called multivectors, see "Multivector" and "Exterior algebra" in Wikipedia.

[2] Not unexpectedly, in order to find $f(x)$ we need not just x but also $\mathbf{n}_x$.

Integrating this over M, we get nothing but the flux! Recall 14c1: the flux of F through M is $\int_M \langle F, \mathbf{n} \rangle$, that is, $\int_{(M,\mathcal{O})} \langle F, \mathbf{n} \rangle \mu_{(M,\mathcal{O})} = \int_{(M,\mathcal{O})} \omega|_M = \int_{(M,\mathcal{O})} \omega$. Well, 14c1 treats a more special case: $M = \partial G$, and $\mathbf{n}$ is directed outwards. Let us generalize it a little.

15d2 Definition. Let $M \subset \mathbb{R}^{n+1}$ be an orientable n-manifold, $F : M \to \mathbb{R}^{n+1}$ a mapping continuous almost everywhere, and $\mathcal{O}$ an orientation of M. The flux of (the vector field) F through (the oriented hypersurface) $(M, \mathcal{O})$ is

$$\int_M \langle F, \mathbf{n} \rangle$$

where $\mathbf{n}$ is the unit normal vector to M that conforms to $\mathcal{O}$. (The integral is treated as improper, and may converge or diverge.)

Thus,

$$\int_{(M,\mathcal{O})} \omega = \int_M \langle F, \mathbf{n} \rangle \tag{15d3}$$

whenever $(M, \mathcal{O})$ is an oriented hypersurface, $\mathbf{n}$ conforms to $\mathcal{O}$, and F corresponds to ω according to (15d1).

Recall 15c4–15c6.

15d4 Exercise. For a 1-form $\omega = f(x, y)\,dx + g(x, y)\,dy$ on $\mathbb{R}^2$ (or an open subset of $\mathbb{R}^2$) prove that the corresponding vector field is $F = (F_1, F_2) = (g, -f)$, and $d\omega = (\operatorname{div} F)\mu_2$.

15d5 Exercise. For the form $\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ find the corresponding vector field F and prove that F is the gradient of the radial harmonic function (14d9).

15d6 Exercise. For the form $\omega = \frac{-y\,dx + x\,dy}{2}$ on $\mathbb{R}^2$ find the corresponding vector field F. Is F the gradient of some function? Of some harmonic function? Find the flux of F through the boundary of the triangle from 11d2.

15d7 Exercise. On $\mathbb{R}^3 \setminus \{0\}$, consider the gradient F of the radial harmonic function (14d9) (for $c_1 = 1$, $c_2 = 0$), and the corresponding 2-form ω. Find the integral of ω over the sphere $\{x : |x| = r\}$.

15e Boundary

We want to apply the divergence theorem to the open cube $B = (0,1)^N$, but for now we cannot, since the boundary ∂B is not a manifold. Rather, ∂B consists of $2N$ disjoint cubes of dimension $n = N-1$ ("hyperfaces") and a finite number[1] of cubes of dimensions $0, 1, \dots, n-1$.

For example, $\{1\} \times (0,1)^n$ is a hyperface.

Each hyperface is an n-manifold, and has exactly two orientations. Also, the outward unit normal vector $\mathbf{n}_x$ is well-defined at every point x of a hyperface.

For example, $\mathbf{n}_x = e_1$ for every $x \in \{1\} \times (0,1)^n$.

For a function f on ∂B we define $\int_{\partial B} f$ as the sum of integrals over the $2N$ hyperfaces; that is,

$$\int_{\partial B} f = \sum_{i=1}^{N} \sum_{x_i=0,1} \underset{(0,1)^n}{\int \cdots \int} f(x_1, \dots, x_N) \prod_{j:j\ne i} \mathrm{d}x_j \,, \tag{15e1}$$

provided that these integrals are well-defined, of course.

For a vector field $F \in C(\partial B \to \mathbb{R}^N)$ we define the flux of F through ∂B as $\int_{\partial B} \langle F, \mathbf{n} \rangle$. Note that

$$\int_{\partial B} \langle F, \mathbf{n} \rangle = \sum_{i=1}^{N} \sum_{x_i=0,1} (2x_i - 1) \underset{(0,1)^n}{\int \cdots \int} F_i(x_1, \dots, x_N) \prod_{j:j\ne i} \mathrm{d}x_j \,. \tag{15e2}$$

It is surprisingly easy to prove the divergence theorem for the cube. (Just from scratch; no need to use 14c3, nor 13b9.)

15e3 Proposition. Let $F \in C^1\big((0,1)^N \to \mathbb{R}^N\big)$, with DF bounded. Then the integral of $\operatorname{div} F$ over $(0,1)^N$ is equal to the (outward) flux of F through the boundary.

(As before, boundedness of DF ensures that F extends to $[0,1]^N$ by continuity; recall 14c4.)

Proof.

$$\int_0^1 D_1 F_1(x_1, \dots, x_N) \, \mathrm{d}x_1 = F_1(1, x_2, \dots, x_N) - F_1(0, x_2, \dots, x_N) = \\ = \sum_{x_1=0,1} (2x_1 - 1) F_1(x_1, \dots, x_N) \,;$$

[1] In fact, $3^N - 1 - 2N$.

$$\int\limits_{(0,1)^N}\!\!\cdots\!\int D_1 F_1 = \sum_{x_1=0,1} (2x_1 - 1) \int\limits_{(0,1)^n}\!\!\cdots\!\int F_1(x_1,\dots,x_N)\,\mathrm{d}x_2\dots\mathrm{d}x_N\,;$$

similarly, for each $i = 1,\dots,N$,

$$\int\limits_{(0,1)^N}\!\!\cdots\!\int D_i F_i = \sum_{x_i=0,1} (2x_i - 1) \int\limits_{(0,1)^n}\!\!\cdots\!\int F_i \prod_{j:j\ne i} \mathrm{d}x_j\ ;$$

it remains to sum over i. □

The same holds for every box B, of course.

Let a vector field F correspond to an n-form ω according to (15d1). We want to think of the flux $\int_{\partial B}\langle F,\mathbf{n}\rangle$ as $\int_{\partial B}\omega$; for now we cannot, since ∂B is not an n-manifold, nor an n-chain. However, we may treat B as a singular N-box $\Gamma : B \to \mathbb{R}^N$, $\Gamma(x) = x$, and then, according to Sections 15b, 15c, ∂B may be treated as an n-chain in two cases, $N = 1$ and $N = 2$. Here is the corresponding construction for arbitrary N.

The $2N$ hyperfaces of $(0,1)^N$ are

$$H_{i,a} = \{(x_1,\dots,x_N) \in [0,1]^n : x_i = a\} \quad \text{for } i = 1,\dots,N \text{ and } a = 0,1\,.$$

For each hyperface $H_{i,a}$ we choose the orientation $\mathcal{O}_{i,a}$ that conforms to $\mathbf{n}_x$ in the sense introduced above: $\mathbf{n}_x = h_1 \times \dots \times h_n$ for some (therefore, every) orthonormal basis $(h_1,\dots,h_n)$ of the tangent space (to the hyperface) that conforms to $\mathcal{O}_{i,a}$. Note that $\mathbf{n}_x = h_1 \times \dots \times h_n$ means $\det(\mathbf{n}_x, h_1,\dots,h_n) = +1$.

Denoting by $(e_1,\dots,e_N)$ the usual basis of $\mathbb{R}^N$, we try the basis $(e_1,\dots,e_{i-1},e_{i+1},\dots,e_N)$ of the tangent space $\{x : x_i = 0\}$ to $H_{i,a}$. We observe that $\det(e_i,e_1,\dots,e_{i-1},e_{i+1},\dots,e_N) = (-1)^{i-1}$, $\mathbf{n}_x = (2a-1)e_i$, and conclude that the basis $(e_1,\dots,e_{i-1},e_{i+1},\dots,e_N)$ conforms to $\mathcal{O}_{i,a}$ if and only if $(-1)^{i-1}(2a-1) = +1$. Thus, the n-chart $\Delta_{i,a}$ of $H_{i,a}$ defined by

$$\Delta_{i,a}(u_1,\dots,u_n) = (u_1,\dots,u_{i-1},a,u_i,\dots,u_n) \quad \text{for } u \in (0,1)^n$$

conforms to $\mathcal{O}_{i,a}$ if and only if $(-1)^{i-1}(2a-1) = +1$. Treating each $\Delta_{i,a}$ (extended to $[0,1]^n$) as a singular n-cube, we define the n-chain ∂B as follows:

$$\partial B = \sum_{i=1}^N \sum_{a=0,1} (-1)^{i-1}(2a-1)\Delta_{i,a}\,. \tag{15e4}$$

Now we have

$$\int_{\partial B}\omega = \int_{\partial B}\langle F,\mathbf{n}\rangle \tag{15e5}$$

whenever ω and F are related via (15d1).[1] This equality results from

$$(-1)^{i-1}(2a-1)\int_{\Delta_{i,a}} \omega = \int_{(H_{i,a},\mathcal{O}_{i,a})} \omega = \int_{H_{i,a}} \langle F, \mathbf{n} \rangle$$

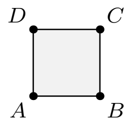
by summation over i and a.

For a singular cube $\Gamma : [0,1]^N \to \mathbb{R}^m$ we define $\partial\Gamma$ as the n-chain

$$\partial\Gamma = \sum_{i=1}^{N} \sum_{a=0,1} (-1)^{i-1}(2a-1)\Gamma \circ \Delta_{i,a} \, . \tag{15e6}$$

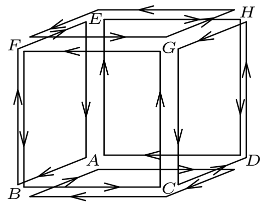
Note that (15e4) is the special case for $\Gamma(x) = x$.

Here is what we get for $N = 2$ and $N = 3$:



$$\Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA} = \Gamma|_{AB} + \Gamma|_{BC} - \Gamma|_{DC} - \Gamma|_{AD} \, ,$$

$$\begin{aligned} &\Gamma|_{ADCB} + \Gamma|_{EFGH} + \Gamma|_{ABFE} + \\ &\quad + \Gamma|_{DHGC} + \Gamma|_{AEHD} + \Gamma|_{BCGF} = \\ &= -\Gamma|_{ABCD} + \Gamma|_{EFGH} - \Gamma|_{AEFB} + \\ &\quad + \Gamma|_{DHGC} - \Gamma|_{ADHE} + \Gamma|_{BCGF} \, . \end{aligned}$$



A cube is only one example of a bounded regular open set $G \subset \mathbb{R}^{n+1}$ such that ∂G is not an n-manifold and still, the divergence theorem holds as $\int_G \operatorname{div} F = \int_{\partial G \setminus Z} \langle F, \mathbf{n} \rangle$ for some closed set $Z \subset \partial G$ such that $\partial G \setminus Z$ is an n-manifold. In such cases we'll say that the divergence theorem holds for G and $\partial G \setminus Z$. For the cube, $\partial G \setminus Z$ is the union of the $2N$ hyperfaces, and Z is the union of cubes of smaller (than $N-1$) dimensions.

15e7 Exercise (PRODUCT)**.** Let $G_1 \subset \mathbb{R}^{N_1}$, $Z_1 \subset \partial G_1$, and $G_2 \subset \mathbb{R}^{N_2}$, $Z_2 \subset \partial G_2$. If the divergence theorem holds for G_1, $\partial G_1 \setminus Z_1$ and for G_2, $\partial G_2 \setminus Z_2$, then it holds for G, $\partial G \setminus Z$ where $G = G_1 \times G_2 \subset \mathbb{R}^{N_1+N_2}$ and $\partial G \setminus Z = \big((\partial G_1 \setminus Z_1) \times G_2 \big) \uplus \big(G_1 \times (\partial G_2 \setminus Z_2) \big)$.

Prove it.[2]

An N-box is the product of N intervals, of course. Also, a cylinder $\{(x,y,z) : x^2 + y^2 < r^2, 0 < z < a\}$ is the product of a disk and an interval.

[1] $\int_{\partial B} \omega$ is the integral of the n-form ω over the n-chain ∂B defined by (15e4); $\int_{\partial B} \langle F, \mathbf{n} \rangle$ is the flux defined by (15e2).

[2] Hint: $\operatorname{div} F = (D_1 F_1 + \dots + D_{N_1} F_{N_1}) + (D_{N_1+1} F_{N_1+1} + \dots + D_{N_1+N_2} F_{N_1+N_2})$.

15f Exterior derivative

In order to find the formula for the exterior derivative $d\omega$ of a form of arbitrary order, we could generalize the approach of Sect. 15c. However, a shorter way is available, via divergence.

Let ω be a $(k-1)$-form on $\mathbb{R}^N$. Assuming existence of a k-form $d\omega$ on $\mathbb{R}^N$ such that $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$ for all singular k-boxes Γ, we want to find $d\omega(x, h_1, \dots, h_k)$. It is sufficient to find $d\omega(x, e_{i_1}, \dots, e_{i_k})$ for $1 \le i_1 < \dots < i_k \le N$; here $(e_1, \dots, e_N)$ is the usual basis of $\mathbb{R}^N$. Let us find $d\omega(x, e_1, \dots, e_k)$; other cases are similar.

Vectors $e_1, \dots, e_k$ span the k-dimensional subspace $\{x : x_{k+1} = \dots = x_N = 0\} = \mathbb{R}^k \subset \mathbb{R}^N$. We need only the restriction $\omega|_{\mathbb{R}^k}$, and re-denote this restriction by ω.

Being a $(k-1)$-form on $\mathbb{R}^k$, the form ω corresponds to a vector field $F : \mathbb{R}^k \to \mathbb{R}^k$ according to (15d1).

For every cube $B \subset \mathbb{R}^k$, by 15d3 and 15e3, $\int_{\partial B} \omega = \int_{\partial B} \langle F, \mathbf{n} \rangle = \int_B \operatorname{div} F$.

Being a k-form on $\mathbb{R}^k$, the form $d\omega$ is $f\mu_k$ for some $f \in C(\mathbb{R}^k)$; here μ_k is the volume form on $\mathbb{R}^k$. Thus, $\int_B d\omega = \int_B f$. The needed equality $\int_B d\omega = \int_{\partial B} \omega$ becomes $\int_B f = \int_B \operatorname{div} F$ (for all B), that is, $f = \operatorname{div} F$. It remains to express this equality in terms of ω and $d\omega$.

We have

$$F_1(x) = \langle F(x), e_1 \rangle = \langle F(x), e_2 \times \dots \times e_k \rangle = \omega(x, e_2, \dots, e_k)\,;$$
$$F_2(x) = \langle F(x), e_2 \rangle = \langle F(x), -e_1 \times e_3 \times \dots \times e_k \rangle = -\omega(x, e_1, e_3, \dots, e_k)\,;$$

and so on. Hence,

$$\begin{aligned} &\operatorname{div} F = D_1 F_1 + \dots + D_k F_k = \\ &= D_1\omega(\cdot, e_2, \dots, e_k) - D_2\omega(\cdot, e_1, e_3, \dots, e_k) + \dots + (-1)^{k-1} D_k \omega(\cdot, e_1, \dots, e_{k-1})\,. \end{aligned}$$

On the other hand,

$$d\omega(x, e_1, \dots, e_k) = f(x)\mu_k(e_1, \dots, e_k) = f(x) = \operatorname{div} F(x)\,.$$

Finally,

$$\begin{aligned} &d\omega(\cdot, e_1, \dots, e_k) = \\ &= D_1\omega(\cdot, e_2, \dots, e_k) - D_2\omega(\cdot, e_1, e_3, \dots, e_k) + \dots + (-1)^{k-1} D_k \omega(\cdot, e_1, \dots, e_{k-1})\,. \end{aligned}$$

The same holds for $e_{i_1}, \dots, e_{i_k}$, and moreover, for arbitrary $h_1, \dots, h_k \in \mathbb{R}^N$, since both sides of this equality are antisymmetric multilinear forms.

15f1 Definition. The *exterior derivative* of a $(k-1)$-form ω of class C^1 is the k-form $d\omega$ defined by

$$(d\omega)(\cdot, h_1, \dots, h_k) = \sum_{i=1}^{k} (-1)^{i-1} D_{h_i}\omega(\cdot, h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_k)\,.$$

For an n-form ω on $\mathbb{R}^N$, $N = n+1$, and $B = [0,1]^N$, we have $d\omega = (\operatorname{div} F)\mu_N$, thus, $\int_B d\omega = \int_B \operatorname{div} F$, whence

$$\int_B d\omega = \int_{\partial B} \omega \tag{15f2}$$

for all n-forms ω on $\mathbb{R}^N$, which is Stokes' theorem for nonsingular cubes.

15f3 Theorem. (*Stokes' theorem*)
Let C be a k-chain in $\mathbb{R}^N$, and ω a $(k-1)$-form of class C^1 on $\mathbb{R}^N$. Then

$$\int_C d\omega = \int_{\partial C} \omega\,.$$

(To be proved later, in Sect. 16d.)

15f4 Exercise. The divergence theorem holds for $G \subset \mathbb{R}^{n+1}$ and $\partial G \setminus Z$ (recall 15e7 and the paragraph before it) if and only if $\int_G d\omega = \int_{\partial G \setminus Z} \omega$ for all n-forms ω on $\mathbb{R}^{n+1}$.
Prove it.

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