16 Stokes' theorem

The ultimate theorem about integral of derivative, Stokes' theorem is the general fundamental theorem of integral calculus.

16a Change of variables

Given a mapping $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^n)$, every singular k-box $\Gamma : B \to \mathbb{R}^\ell$ leads to a singular k-box $\varphi \circ \Gamma : B \to \mathbb{R}^n$. Thus, every k-form ω on $\mathbb{R}^n$ leads to a box function $\Gamma \mapsto \int_{\varphi\circ\Gamma} \omega$; it is additive (since the mapping $\Gamma \mapsto \varphi \circ \Gamma$ is). Can we find a k-form $\varphi^*\omega$ on $\mathbb{R}^\ell$ such that $\int_{\varphi\circ\Gamma} \omega = \int_\Gamma \varphi^*\omega$ for all Γ?

16a1 Definition. Given a k-form ω on $\mathbb{R}^n$ and a mapping $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^n)$, the *pullback* of ω along φ is a k-form $\varphi^*\omega$ on $\mathbb{R}^\ell$ defined by

$$(\varphi^*\omega)(x, h_1, \dots, h_k) = \omega\big(\varphi(x), (D\varphi)_x(h_1), \dots, (D\varphi)_x(h_k)\big) = \\ = \omega\big(\varphi(x), (D_{h_1}\varphi)_x, \dots, (D_{h_k}\varphi)_x\big) \quad \text{for } x, h_1, \dots, h_k \in \mathbb{R}^\ell .$$

The form $\varphi^*\omega$ is of class C^m whenever ω is of class C^m and φ is of class C^{m+1}. The mapping $\omega \mapsto \varphi^*\omega$ is linear. For $k = 0$ the pullback is just the composition: $(\varphi^* f)(x) = f(\varphi(x))$; $\varphi^* f = f \circ \varphi$ (no need in C^{m+1} in this case). And $\varphi^*(f\omega) = (\varphi^* f)(\varphi^*\omega) = (f \circ \varphi)\varphi^*\omega$ for $f \in C^1(\mathbb{R}^n)$.

16a2 Lemma. $(\psi\circ\varphi)^*\omega = \varphi^*(\psi^*\omega)$ for all $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^m)$, $\psi \in C^1(\mathbb{R}^m \to \mathbb{R}^n)$, and k-forms ω on $\mathbb{R}^n$.

Proof. By the chain rule 2b11,

$$\big(D(\psi \circ \varphi)\big)_x = (D\psi)_{\varphi(x)} \circ (D\varphi)_x ;$$

thus,

$$\big((\psi\circ\varphi)^*\omega\big)(x, h_1, \dots, h_k) = \omega\big((\psi\circ\varphi)(x), (D(\psi\circ\varphi))_x(h_1), \dots, (D(\psi\circ\varphi))_x(h_k)\big) =$$

$$= \omega\big(\psi(\varphi(x)), (D\psi)_{\varphi(x)}(D\varphi)_x h_1, \dots, (D\psi)_{\varphi(x)}(D\varphi)_x h_k\big) =$$
$$= (\psi^*\omega)\big(\varphi(x), (D\varphi)_x h_1, \dots, (D\varphi)_x h_k\big) = \big(\varphi^*(\psi^*\omega)\big)(x, h_1, \dots, h_k)\,.$$

□

The same applies to open subsets of $\mathbb{R}^\ell, \mathbb{R}^m, \mathbb{R}^n$, of course.

A singular k-box Γ in $\mathbb{R}^n$ is a C^1-mapping $B \to \mathbb{R}^n$ on a box $B \subset \mathbb{R}^k$; the pullback $\Gamma^*\omega$ is well-defined,

$$(\Gamma^*\omega)(u, h_1, \dots, h_k) = \omega\big(\Gamma(u), (D_{h_1}\Gamma)_u, \dots, (D_{h_k}\Gamma)_u\big)$$

for $u \in B^\circ$ and $h_1, \dots, h_k \in \mathbb{R}^k$. As every k-form on $\mathbb{R}^k$, $\Gamma^*\omega$ is $f\mu_k$, where μ_k is the volume form on $\mathbb{R}^k$, and $f(u) = (\Gamma^*\omega)(u, e_1, \dots, e_k) = \omega\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)$. Thus, $\int_B \Gamma^*\omega = \int_B f =$ $\int_B \omega\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)\,\mathrm{d}u$. It means that the definition (11e12) of $\int_\Gamma \omega$ may be rewritten as

(16a3) $$\int_\Gamma \omega = \int_B \Gamma^*\omega\,.$$

We see that it was the integral of the pullback, from the very beginning!

Let Γ be a singular k-box in $\mathbb{R}^\ell$, $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^n)$, and ω a k-form on $\mathbb{R}^n$. By 16a2, $(\varphi \circ \Gamma)^*\omega = \Gamma^*(\varphi^*\omega)$ on B°; integrating this we get the change of variable formula

(16a4) $$\int_{\varphi\circ\Gamma} \omega = \int_\Gamma \varphi^*\omega$$

for singular boxes, and therefore (by linearity in C), also for k-chains C in $\mathbb{R}^n$:

(16a5) $$\int_{\varphi\circ C} \omega = \int_C \varphi^*\omega\,,$$

where $\varphi \circ C = c_1(\varphi \circ \Gamma_1) + \dots + c_p(\varphi \circ \Gamma_p)$ for $c = c_1\Gamma_1 + \dots + c_p\Gamma_p$. In particular, $\partial\Gamma$ is a $(k-1)$-chain, and $\varphi \circ \partial\Gamma = \partial(\varphi \circ \Gamma)$, since (recall (15e6))

$$\varphi \circ \partial\Gamma = \varphi \circ \Big(\sum_{i=1}^k \sum_{a=0,1} (-1)^{i-1}(2a-1)\Gamma \circ \Delta_{i,a} \Big) =$$
$$= \sum_{i=1}^k \sum_{a=0,1} (-1)^{i-1}(2a-1)\varphi \circ \Gamma \circ \Delta_{i,a} = \partial(\varphi \circ \Gamma)\,;$$

for this chain (16a5) gives

(16a6) $$\int_{\partial(\varphi\circ\Gamma)} \omega = \int_{\partial\Gamma} \varphi^*\omega\,.$$

16b A special case of differential form

Given a mapping $\varphi \in C^1(\mathbb{R}^n \to \mathbb{R}^k)$, $\varphi(x) = \big(\varphi_1(x), \dots, \varphi_k(x)\big)$, we denote[1]

$$d\varphi_1 \wedge \dots \wedge d\varphi_k = \varphi^* \mu_k \,, \tag{16b1}$$

where μ_k is the volume form on $\mathbb{R}^k$. That is,

$$\begin{aligned}(d\varphi_1 \wedge \dots \wedge d\varphi_k)(x, h_1, \dots, h_k) &= \mu_k\big(\varphi(x), (D\varphi)_x h_1, \dots, (D\varphi)_x h_k\big) = \\ &= \det\big((D\varphi)_x h_1, \dots, (D\varphi)_x h_k\big) = \det\big((D_{h_i}\varphi_j)_x\big)_{i,j} \,.\end{aligned}$$

The last determinant shows that the mapping $(\varphi_1, \dots, \varphi_k) \mapsto d\varphi_1 \wedge \dots \wedge d\varphi_k$ is antisymmetric and multilinear; that is,

$$\begin{gathered} d\varphi_i \wedge d\varphi_1 \wedge \dots \wedge d\varphi_{i-1} \wedge d\varphi_{i+1} \dots \wedge d\varphi_k = (-1)^{i-1} d\varphi_1 \wedge \dots \wedge d\varphi_k \,, \\ d(\varphi_1 + \psi_1) \wedge d\varphi_2 \wedge \dots \wedge d\varphi_k = d\varphi_1 \wedge \dots \wedge d\varphi_k + d\psi_1 \wedge d\varphi_2 \wedge \dots \wedge d\varphi_k \,. \end{gathered}$$

If $\varphi \in C^2(\mathbb{R}^n \to \mathbb{R}^k)$, that is, $\varphi_1, \dots, \varphi_k \in C^2(\mathbb{R}^n)$, then $d\varphi_1 \wedge \dots \wedge d\varphi_k$ is of class C^1.

In particular (for $n = k$, $\varphi = \mathrm{id}$), using the informal but habitual notation x_i for the function $(x_1, \dots, x_n) \mapsto x_i$, we have

$$\begin{gathered} dx_1 \wedge \dots \wedge dx_n = \mu_n \,; \\ (dx_1 \wedge \dots \wedge dx_n)(x, h_1, \dots, h_n) = \det(h_1, \dots, h_n) \,. \end{gathered} \tag{16b2}$$

The case $k = n - 1$ is of special interest (as before).

16b3 Lemma. For arbitrary $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}^{n+1}$,

$$\det\big(\langle a_i, b_j \rangle\big)_{i,j} = \langle a_1 \times \dots \times a_n, b_1 \times \dots \times b_n \rangle \,.$$

Proof. Both sides of this formula are antisymmetric multilinear n-forms in $a_1, \dots, a_n$ (for given $b_1, \dots, b_n$). Thus, WLOG, $a_1 = e_{p_1}, \dots, a_n = e_{p_n}$ for some $1 \le p_1 < \dots < p_n \le n+1$. Similarly, $b_1 = e_{q_1}, \dots, b_n = e_{q_n}$ for some $1 \le q_1 < \dots < q_n \le n+1$. Now, both sides equal 1 if $p_1 = q_1, \dots, p_n = q_n$, otherwise 0. □

16b4 Lemma. For every $\varphi \in C^1(\mathbb{R}^{n+1} \to \mathbb{R}^n)$, the n-form $d\varphi_1 \wedge \dots \wedge d\varphi_n$ corresponds, according to (15d1), to the vector field $F : x \mapsto \nabla\varphi_1(x) \times \dots \times \nabla\varphi_n(x)$.

[1] In fact, it is possible to define the corresponding associative binary operation (so-called exterior product) $\omega_1, \omega_2 \mapsto \omega_1 \wedge \omega_2$ (a $(k+l)$-form, if ω_1 is a k-form and ω_2 is an l-form).

Proof. $(d\varphi_1\wedge\dots\wedge d\varphi_n)(x,h_1,\dots,h_n) = \det\big((D_{h_i}\varphi_j)_x\big)_{i,j} = \det\big(\langle\nabla\varphi_j(x),h_i\rangle\big)_{i.j} = \langle\nabla\varphi_1(x)\times\dots\times\nabla\varphi_n(x), h_1\times\dots\times h_n\rangle$. □

Note that $(D_{F(x)}\varphi)_x = 0$.

We return to arbitrary k and n. A bit more generally than (16b2), for a linear $\varphi : \mathbb{R}^n \to \mathbb{R}^k$, we have $D_h\varphi = \varphi(h)$, thus,

$$(d\varphi_1 \wedge \dots \wedge d\varphi_k)(h_1,\dots,h_k) = \det\big(\varphi(h_1),\dots,\varphi(h_k)\big) = \det\big((\varphi_i(h_j))_{i,j}\big)$$

irrespective of x; not depending on x, this $d\varphi_1\wedge\dots\wedge d\varphi_k$ may be interpreted not only as a differential form, but also as an antisymmetric multilinear form. In particular, given $1 \le m_1 < \dots < m_k \le n$,

$$(dx_{m_1} \wedge \dots \wedge dx_{m_k})(h_1,\dots,h_k) = \det\big(\langle h_j, e_{m_i}\rangle\big)_{i,j}$$

is a minor of the matrix $(h_1,\dots,h_k)$ corresponding to the rows $m_1,\dots,m_k$.

16b5 Lemma. For every antisymmetric multilinear k-form L on $\mathbb{R}^n$,

$$L = \sum_{1\le m_1<\dots<m_k\le n} L(e_{m_1},\dots,e_{m_k})\, dx_{m_1}\wedge\dots\wedge dx_{m_k}\,.$$

Proof. Both sides of this formula are antisymmetric multilinear k-forms; we have to prove that they are equal on arbitrary $h_1,\dots,h_k \in \mathbb{R}^n$. WLOG, $h_1 = e_{p_1},\dots,h_k = e_{p_k}$ for some $1 \le p_1 < \dots < p_k \le n$. It remains to note that

$$(dx_{m_1}\wedge\dots\wedge dx_{m_k})(e_{p_1},\dots,e_{p_k}) = \begin{cases} 1 & \text{if } m_1 = p_1,\dots,m_k = p_k, \\ 0 & \text{otherwise.}\end{cases}$$

□

It follows that for every (differential) k-form ω on $\mathbb{R}^n$,

$$\omega = \sum_{1\le m_1<\dots<m_k\le n} f_{m_1,\dots,m_k}(x)\, dx_{m_1}\wedge\dots\wedge dx_{m_k}\,, \qquad f_{m_1,\dots,m_k}(x) = \omega(x, e_{m_1},\dots,e_{m_k})\,. \tag{16b6}$$

Let $\varphi \in C^1(\mathbb{R}^n \to \mathbb{R}^m)$ and $\psi \in C^1(\mathbb{R}^m \to \mathbb{R}^k)$; by Lemma 16a2, $(\psi\circ\varphi)^*\mu_k = \varphi^*(\psi^*\mu_k)$, hence, $d(\psi\circ\varphi)_1\wedge\dots\wedge d(\psi\circ\varphi)_k = \varphi^*(d\psi_1\wedge\dots\wedge d\psi_k)$, that is,

$$\varphi^*(d\psi_1\wedge\dots\wedge d\psi_k) = d(\psi_1\circ\varphi)\wedge\dots\wedge d(\psi_k\circ\varphi))\,; \tag{16b7}$$

and therefore,

$$\varphi^*(f\, d\psi_1\wedge\dots\wedge d\psi_k) = (f\circ\varphi)\, d(\psi_1\circ\varphi)\wedge\dots\wedge d(\psi_k\circ\varphi)) \tag{16b8}$$

for $f \in C^1(\mathbb{R}^n)$.

16b9 Lemma. If $\omega = f\,dx_{m_1} \wedge \dots \wedge dx_{m_k}$, then $d\omega = df \wedge dx_{m_1} \wedge \dots \wedge dx_{m_k}$.

Proof. We apply both forms to $e_m, e_{m_1}, \dots, e_{m_k}$ for arbitrary $m \in \{1, \dots, n\} \setminus \{m_1, \dots, m_k\}$. First, by 15f1,

$$(d\omega)(\cdot, e_m, e_{m_1}, \dots, e_{m_k}) = D_m \omega(\cdot, e_{m_1}, \dots, e_{m_k}) = D_m f\,,$$

since the other terms (for $i = 2, \dots, k$) in 15f1 vanish. Second,

$$(df \wedge dx_{m_1} \wedge \dots \wedge dx_{m_k})(\cdot, e_m, e_{m_1}, \dots, e_{m_k}) = \left(\begin{array}{c|c} D_m f & D_{m_1} f \dots D_{m_k} f \\ \hline 0 & I \end{array}\right) = D_m f\,.$$

Finally, both forms vanish unless $e_{m_1}, \dots, e_{m_k}$ are present among the $k+1$ chosen basis vectors. $\square$

16b10 Exercise. $df \wedge dx_{m_1} \wedge \dots \wedge dx_{m_k} = \sum_i (D_i f) dx_i \wedge dx_{m_1} \wedge \dots \wedge dx_{m_k}$. Prove it.[1]

16c Stokes' theorem for the volume form

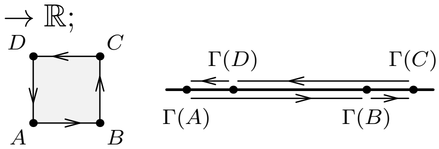
16c1 Proposition. $\int_{\partial\Gamma} \mu_n = 0$ for every singular $(n+1)$-box in $\mathbb{R}^n$.

Here μ_n is the volume form on $\mathbb{R}^n$, that is, $\mu_n(x, h_1, \dots, h_n) = \det(h_1, \dots, h_n)$. Clearly, $d\mu_n = 0$ (since the determinant does not depend on x), thus, $\int_\Gamma d\mu_n = 0$, and 16c1 is a case of Stokes' theorem 15f3.

16c2 Example. $n = 0$; $\mathbb{R}^n = \{0\}$, $\Gamma : [0,1] \to \{0\}$, $\partial\Gamma = \{0\} - \{0\} = 0$.

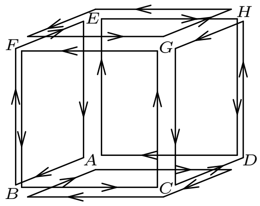
16c3 Example. $n = 1$; $\Gamma : [0,1] \times [0,1] \to \mathbb{R}$;

$$\partial\Gamma = \Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA}\,;$$
$$\int_{\Gamma|_{AB}} \mu_1 = \Gamma(B) - \Gamma(A)\,;$$



the four signed lengths sum up to 0.

16c4 Example. $n = 2$; $\Gamma : [0,1]^3 \to \mathbb{R}^2$;



look twice: (a) see the 3-dimensional cube; (b) see its planar image, and note that the six signed areas sum up to 0.

16c5 Lemma. It is sufficient to prove Prop. 16c1 for $\Gamma \in C^2(B \to \mathbb{R}^n)$.

[1] Hint: follow the spirit of the proof of 16b9.

Proof. It is sufficient to prove that $C^2(B \to \mathbb{R}^n)$ is dense in $C^1(B \to \mathbb{R}^n)$; that is, for arbitrary $\Gamma \in C^1(B \to \mathbb{R}^n)$ and $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in C^2(B \to \mathbb{R}^n)$ such that, for all $u \in B^\circ$, $|\Gamma_\varepsilon(u) - \Gamma(u)| \le \varepsilon$ and $|(D\Gamma_\varepsilon)_u - (D\Gamma)_u| \le \varepsilon$; then $|\int_{\partial\Gamma_\varepsilon} \mu_n - \int_{\partial\Gamma} \mu_n| = \mathcal{O}(\varepsilon)$.

Here is the proof for $B = [0,1] \times [0,1] \subset \mathbb{R}^2$ (the general case is similar; see also 7d27, 7d28, 7e3).

We define Γ_ε by

$$\Gamma_\varepsilon(u_1, u_2) = \frac{1}{\varepsilon^2} \int_{[u_1,u_1+\varepsilon]\times[u_2,u_2+\varepsilon]} \Gamma\Big(\frac{v_1}{1+\varepsilon}, \frac{v_2}{1+\varepsilon}\Big)\, \mathrm{d}v_1 \mathrm{d}v_2 \,,$$

then the partial derivative

$$\frac{\partial}{\partial u_1}\Gamma_\varepsilon(u_1, u_2) = \frac{1}{\varepsilon} \int_{[u_2,u_2+\varepsilon]} \frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{u_1+\varepsilon}{1+\varepsilon}, \frac{v_2}{1+\varepsilon}\Big) - \Gamma\Big(\frac{u_1}{1+\varepsilon}, \frac{v_2}{1+\varepsilon}\Big)\Big)\, \mathrm{d}v_2$$

is of class C^1 and converges (uniformly) to $\frac{\partial}{\partial u_1}\Gamma(u_1, u_2)$. □

Proof *of Prop. 16c1 for* $n = 2$.
By 16c5, WLOG, $\Gamma \in C^2(B \to \mathbb{R}^2)$, $B = [0,1]^3$. By (16a6) applied to $\Gamma \circ \mathrm{id}$, $\int_{\partial\Gamma} \mu_2 = \int_{\partial B} \Gamma^*\mu_2$. The 2-form $\Gamma^*\mu_2 = d\Gamma_1 \wedge d\Gamma_2$ of class C^1 on B corresponds, by 16b4, to the vector field $F \in C^1(B \to \mathbb{R}^3)$,

$$F(u) = \nabla\Gamma_1(u) \times \nabla\Gamma_2(u)\,.$$

By (15e5) and 15e3, $\int_{\partial B} \Gamma^*\mu_2 = \int_{\partial B} \langle F, \mathbf{n}\rangle = \int_B \operatorname{div} F$. It remains to prove that $\operatorname{div} F = 0$.[1]

We have

$$F_1 = \det(D_2\Gamma, D_3\Gamma)\,, \quad F_2 = -\det(D_1\Gamma, D_3\Gamma)\,, \quad F_3 = \det(D_1\Gamma, D_2\Gamma)\,,$$

(since $F_j = \langle e_j, \nabla\Gamma_1 \times \nabla\Gamma_2\rangle = \det(e_j, \nabla\Gamma_1, \nabla\Gamma_2)$), thus

$$\begin{aligned}
\operatorname{div} F &= D_1F_1 + D_2F_2 + D_3F_3 = \\
&= \det(D_1D_2\Gamma, D_3\Gamma) + \det(D_2\Gamma, D_1D_3\Gamma) - \\
&\quad - \det(D_2D_1\Gamma, D_3\Gamma) - \det(D_1\Gamma, D_2D_3\Gamma) + \\
&\quad + \det(D_3D_1\Gamma, D_2\Gamma) + \det(D_1\Gamma, D_3D_2\Gamma) = \\
&\qquad = \det(D_1D_2\Gamma, D_3\Gamma) - \det(D_2D_1\Gamma, D_3\Gamma) + \\
&\qquad + \det(D_2\Gamma, D_1D_3\Gamma) + \det(D_3D_1\Gamma, D_2\Gamma) - \\
&\qquad - \det(D_1\Gamma, D_2D_3\Gamma) + \det(D_1\Gamma, D_3D_2\Gamma) = 0\,.
\end{aligned}$$

□

[1] Basically, we'll examine an infinitesimal box in quadratic approximation.

Proof *of Prop. 16c1 (in general).*
The first part of the proof for $n = 2$ needs only trivial changes; $\Gamma^*\mu_n$ corresponds to $F \in C^1(B \to \mathbb{R}^n)$, $B = [0,1]^{n+1}$,

$$F(u) = \nabla\Gamma_1(u) \times \dots \times \nabla\Gamma_n(u)\,;$$

we have to prove that $\operatorname{div} F = 0$.

Introducing

$$A_i = \det\bigl(D_1\Gamma, \dots, D_{i-1}\Gamma, D_{i+1}\Gamma, \dots, D_n\Gamma\bigr)\,,$$

$$B_{i,j} = \begin{cases} \det\bigl(D_1\Gamma, \dots, D_{i-1}\Gamma, D_{i+1}\Gamma, \dots, D_{j-1}\Gamma, D_iD_j\Gamma, D_{j+1}\Gamma, \dots, D_n\Gamma\bigr) & \text{for } i < j, \\ \det\bigl(D_1\Gamma, \dots, D_{j-1}\Gamma, D_iD_j\Gamma, D_{j+1}\Gamma, \dots, D_{i-1}\Gamma, D_{i+1}\Gamma, \dots, D_n\Gamma\bigr) & \text{for } j < i, \end{cases}$$

we have

$$F_i = (-1)^{i-1}A_i\,; \quad D_iA_i = \sum_{j:j\ne i} B_{i,j}\,; \quad B_{j,i} = (-1)^{j-i-1}B_{i,j}\,;$$

hence

$$\begin{aligned} \operatorname{div} F = \sum_i D_iF_i = \sum_i (-1)^{i-1} \sum_{j:j\ne i} B_{i,j} = \\ = \sum_{i,j:i\ne j} (-1)^{i-1}B_{i,j} = \sum_{i,j:i<j} \bigl((-1)^{i-1}B_{i,j} + (-1)^{j-1}B_{j,i}\bigr) = 0\,. \end{aligned}$$

□

16d Proving Stokes' theorem (in general)

16d1 Remark. Given $\varphi \in C^1(\mathbb{R}^n \to \mathbb{R}^\ell)$, a $(k-1)$-form ω on $\mathbb{R}^\ell$, and a singular k-box Γ in $\mathbb{R}^n$, we may consider two cases of Stokes' theorem 15f3,

(a) $$\int_\Gamma d(\varphi^*\omega) = \int_{\partial\Gamma} \varphi^*\omega\,,$$

(b) $$\int_{\varphi\circ\Gamma} d\omega = \int_{\partial(\varphi\circ\Gamma)} \omega\,.$$

The change of variables (16a4), (16a6) gives

$$\int_{\varphi\circ\Gamma} d\omega = \int_\Gamma \varphi^*(d\omega)\,, \qquad \int_{\partial(\varphi\circ\Gamma)} \omega = \int_{\partial\Gamma} \varphi^*\omega\,.$$

Thus, we may rewrite (a) and (b) as

$$\int_\Gamma d(\varphi^*\omega) = \int_{\partial(\varphi\circ\Gamma)} \omega \, ,$$
$$\int_\Gamma \varphi^*(d\omega) = \int_{\partial(\varphi\circ\Gamma)} \omega \, .$$

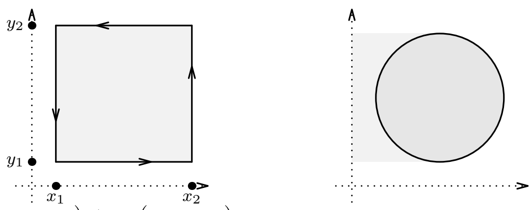
In order to conclude that (a)$\Longleftrightarrow$(b) we need to know that $d(\varphi^*\omega) = \varphi^*(d\omega)$.[1]

16d2 Lemma. In order to obtain Stokes' theorem for all $(k-1)$-forms of class C^1 on $\mathbb{R}^N$, it is sufficient to have it for the $(k-1)$-form

$$\nu_{k-1} = x_1 \, dx_2 \wedge \dots \wedge dx_k$$

on $\mathbb{R}^k$.

16d3 Example. The 1-form ν_1 on $\mathbb{R}^2$ is $x_1 \, dx_2$, that is, $x \, dy$. For every box $B \subset \mathbb{R}^2$, $\int_{\partial B} \nu_1 = \int_B \mu_2 = v(B)$.



$x_2(y_2 - y_1) + x_1(y_1 - y_2) =$
$= (x_2 - x_1)(y_2 - y_1)$

The same holds for every "good" planar domain.
Think, what happens in three dimensions.

***Proof** of Lemma 16d2.* By (16b6), all $(k-1)$-forms of class C^1 on $\mathbb{R}^N$ are linear combinations of such forms:

$$\omega = f(x) \, dx_{m_1} \wedge \dots \wedge dx_{m_{k-1}}$$

for $f \in C^1(\mathbb{R}^N)$ and $1 \le m_1 < \dots < m_{k-1} \le N$. Due to linearity (in ω) of both sides of Stokes' theorem, WLOG, ω is as above. We introduce $\varphi : \mathbb{R}^N \to \mathbb{R}^k$ by $\varphi(x) = \big(f(x), x_{m_1}, \dots, x_{m_{k-1}}\big)$. By (16b8), $\omega = \varphi^*\nu_{k-1}$. By (16b9), $d\nu_{k-1} = \mu_k$ and $d\omega = df \wedge dx_{m_1} \wedge \dots \wedge dx_{m_{k-1}}$; the latter is $\varphi^*\mu_k$ (just by (16b1)). We get

$$d\big(\varphi^*\nu_{k-1}\big) = d\omega = \varphi^*\mu_k = \varphi^*(d\nu_{k-1}) \, .$$

It remains to use Remark 16d1. $\square$

[1]Ultimately we'll see that this holds for all φ and ω; see 16e1.

Proof *of Stokes' theorem 15f3.*
By Lemma 16d2, WLOG, $N = k$ and $\omega = \nu_{k-1}$. Similarly to 16c5 we assume that $\Gamma \in C^2(B \to \mathbb{R}^k)$, $B = [0,1]^k$. Similarly to the proof of Prop. 16c1, $\Gamma^* \nu_{k-1}$ corresponds to a vector field; by (16b8), this vector field is fF where $f = \Gamma_1$ and $F = \nabla\Gamma_2 \times \dots \times \nabla\Gamma_k$. We note that the vector field F is the same as in the proof of Prop. 16c1, but for the singular k-box $u \mapsto \big(\Gamma_2(u), \dots, \Gamma_k(u)\big)$ in $\mathbb{R}^{k-1}$. As was seen there, $\operatorname{div} F = 0$. By 14c5, $\operatorname{div}(fF) = \langle \nabla f, F \rangle$. As before, $\int_{\partial\Gamma} \nu_{k-1} = \int_{\partial B} \Gamma^* \nu_{k-1} = \int_{\partial B} \langle fF, \mathbf{n} \rangle = \int_B \operatorname{div}(fF)$. It remains to prove that $\int_B \operatorname{div}(fF) = \int_\Gamma d\nu_{k-1}$, that is, $\int_B \langle \nabla f, F \rangle = \int_\Gamma \mu_k$.

By (11e12) and the definition of μ_k,

$$\int_\Gamma \mu_k = \int_B \det(D_1\Gamma, \dots, D_k\Gamma) = \int_B \det D\Gamma\,.$$

On the other hand,

$$\langle \nabla f, F \rangle = \langle \nabla\Gamma_1, \nabla\Gamma_2 \times \dots \times \nabla\Gamma_k \rangle = \det(\nabla\Gamma_1, \dots, \nabla\Gamma_k) = \det D\Gamma\,.$$

□

16e Some implications

ON DIFFEOMORPHISM INVARIANCE

16e1 Proposition. $\varphi^*(d\omega) = d(\varphi^*\omega)$ whenever $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^n)$ and ω is a k-form of class C^1 on $\mathbb{R}^n$.

Proof. For every singular k-box Γ in $\mathbb{R}^\ell$,

$$\int_\Gamma \varphi^*(d\omega) = \int_{\varphi\circ\Gamma} d\omega = \int_{\partial(\varphi\circ\Gamma)} \omega = \int_{\partial\Gamma} \varphi^*\omega = \int_\Gamma d(\varphi^*\omega)\,.$$

□

In particular, when $\ell = n$ and φ is a diffeomorphism, we get a one-to-one correspondence between forms (ω and $\varphi^*\omega$), and this correspondence preserves all operations on forms. The calculus of forms is diffeomorphism invariant. Its formulas look the same in all (curvilinear) coordinates!

16e2 Corollary. If $\omega = f_0\, df_1 \wedge \dots \wedge df_k$ then $d\omega = df_0 \wedge df_1 \wedge \dots \wedge df_k$, for arbitrary $f_0, f_1, \dots, f_k \in C^1(\mathbb{R}^n)$.

Proof. We introduce $\varphi : \mathbb{R}^n \to \mathbb{R}^{k+1}$ by $\varphi(x) = \big(f_0(x), \dots, f_k(x)\big)$, note that $\omega = \varphi^* \nu_k$ by (16b8), $d\nu_k = \mu_{k+1}$, $\varphi^* \mu_{k+1} = df_0 \wedge \dots \wedge df_k$ by (16b7); and $\varphi^*(d\nu_k) = d(\varphi^* \nu_k)$ by 16e1. □

16e3 Definition. A form ω of class C^1 is *closed* if $d\omega = 0$.

16e4 Exercise. (a) If $\omega = df_1 \wedge \dots \wedge df_k$ for some $f_1, \dots, f_k \in C^2(\mathbb{R}^n)$, then ω is closed.

(b) For every form ω of class C^2 the form $d\omega$ is closed.
Prove it.

That is, $d(d\omega) = 0$ always.

It is easy to generalize the pullback $\varphi^*\omega$ (defined by 16a1) to a form ω on a manifold $M \subset \mathbb{R}^n$ (rather than the whole $\mathbb{R}^n$) and $\varphi : \mathbb{R}^\ell \to M$. In particular, for a chart (G, ψ) of M the pullback $\psi^*\omega$ is a form on G, and we may define $d\omega$ as a form on M such that $\psi^*(d\omega) = d(\psi^*\omega)$ for all charts. Prop. 16e1 ensures existence of such $d\omega$ via a counterpart of 12a9. Then it is easy to generalize Stokes' theorem to forms (and singular boxes) on M. Still, k-forms on M for $k+1 = \dim M$ correspond to tangent vector fields on M, and the exterior derivative corresponds to divergence (as in Sect. 15f). However, formulas for divergence look differently in different coordinates; they are not diffeomorphism invariant. Also the correspondence between forms and vector fields is not diffeomorphism invariant.

16e5 Exercise. Let $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ be a diffeomorphism, and $G \subset \mathbb{R}^N$, $Z \subset \partial G$ such that the divergence theorem holds for $G, \partial G \setminus Z$. Then it holds also for $\varphi(G), \varphi(\partial G \setminus Z)$.

Prove it.[1]

16e6 Exercise (CONE)**.** Consider in $\mathbb{R}^3$ the cylinder $G_1 = \{(x,y) : x^2+y^2 < 1\} \times (0,1)$, the cone $G_2 = \{(x,y,z) : x^2+y^2 < z^2, 0 < z < 1\}$, and the mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x,y,z) = (xz, yz, z)$.

(a) $\int_{G_1} \varphi^*\omega = \int_{G_2} \omega$ for every 3-form ω on G_2;

(b) $\int_{\partial G_1 \setminus Z_1} \varphi^*\omega = \int_{\partial G_2 \setminus Z_2} \omega$ for every 2-form ω on $\partial G_2 \setminus Z_2$; here $Z_1 = \{(x,y) : x^2+y^2 = 1\} \times \{0,1\} \subset \partial G_1$, $Z_2 = \{(0,0,0)\} \cup \{(x,y) : x^2+y^2 = 1\} \times \{1\} \subset \partial G_2$;

(c) $\int_{G_2} d\omega = \int_{\partial G_2 \setminus Z_2} \omega$ for every 2-form ω of class C^2 on a neighborhood of $\overline{G}_2$;

(d) the divergence theorem holds for $G_2, \partial G_2 \setminus Z_2$.
Prove it.[2]

[1] Hint: $\int_G \varphi^*(d\omega) = \int_{\varphi(G)} d\omega$ and $\int_{\partial G \setminus Z} \varphi^*\omega = \int_{\varphi(\partial G \setminus Z)} \omega$.

[2] Hint: (c) use (a), (b); recall 15e7 and the paragraph after it.

16e7 Exercise (CONE)**.** Let $G \subset \mathbb{R}^n$, $Z \subset \partial G$ be such that the divergence theorem holds for $G, \partial G \setminus Z$. Consider such sets in $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}$:

$$G_1 = G \times (0,1)\,, \qquad Z_1 = (\partial G \times \{0,1\}) \cup (Z \times [0,1])\,,$$
$$G_2 = \{(tx,t) : x \in G, t \in (0,1)\}\,, \qquad Z_2 = \{(0,0)\} \cup (Z \times [0,1])\,.$$

Generalize 16e6 to this situation; prove that the divergence theorem holds for $G_2, \partial G_2 \setminus Z_2$.

16e8 Exercise (SIMPLEX)**.** Using 16e7 and induction in n, obtain the divergence theorem for the simplex $\{(x_1,\dots,x_n) \in (0,\infty)^n : x_1 + \dots + x_n < 1\}$.

ON CONVERGENCE OF SINGULAR BOXES

Recall 15a3: two k-chains C_1, C_2 are equivalent ($C_1 \sim C_2$) if $\int_{C_1} \omega = \int_{C_2} \omega$ for all k-forms ω of class C^0. Or equivalently, of class C^1 (since these are dense).

16e9 Proposition. If $C_1 \sim C_2$ then $\partial C_1 \sim \partial C_2$.

Proof.

$$\int_{\partial C_1} \omega = \int_{C_1} d\omega = \int_{C_2} d\omega = \int_{\partial C_2} \omega\,.$$

□

Now, recall convergence of paths (11b11); equivalently, $\gamma_j \to \gamma$ when there exist $\varepsilon_j \to 0$ and L such that for all $t \in (t_0, t_1)$,

$$|\gamma_j(t) - \gamma(t)| \le \varepsilon_j\,, \quad |\gamma_j'(t)| \le L\,.$$

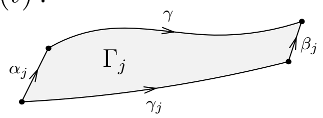
16e10 Proposition. If $\gamma_j \to \gamma$ then $\int_{\gamma_j} \omega \to \int_\gamma \omega$ for every 1-form ω.

16e11 Remark. The condition $|\gamma_j'(t)| \le L$ cannot be dropped. Here is a counterexample:

$$\gamma_j(t) = \frac{1}{\sqrt{j}}(\cos jt, \sin jt) \quad \text{for } t \in [0, 2\pi]\,,$$
$$\gamma_j \to \gamma\,, \quad \gamma(t) = (0,0)\,;$$
$$\omega = x\,dy - y\,dx\,;$$
$$\int_{\gamma_j} \omega = \int_0^{2\pi} \frac{1}{j}\big(\cos jt \cdot (\sin jt)' - \sin jt \cdot (\cos jt)'\big)\,\mathrm{d}t = 2\pi \quad \text{for all } j\,;$$
$$\int_\gamma \omega = 0\,.$$

Proof *(sketch) of Prop. 16e10.*
WLOG, ω is of class C^1 (otherwise, approximate it by $\omega_\delta \in C^1$, $|\omega(x,h) - \omega_\delta(x,h)| \le \delta |h|$, then $|\int_{\gamma_j} (\omega - \omega_\delta)| \le \delta L(t_1 - t_0)$). We take boxes $B_j = [t_0, t_1] \times [0, \varepsilon_j] \subset \mathbb{R}^2$ and define singular 2-boxes $\Gamma_j : B_j \to \mathbb{R}^n$ by

$$\Gamma_j(t,u) = \Big(1 - \frac{u}{\varepsilon_j}\Big)\gamma_j(t) + \frac{u}{\varepsilon_j}\gamma(t) \, .$$



We have $\Gamma_j(\cdot, 0) = \gamma_j$ and $\Gamma_j(\cdot, \varepsilon_j) = \gamma$, thus,

$$\partial \Gamma_j = \gamma_j - \gamma + \beta_j - \alpha_j \, ,$$

$\int_{\alpha_j} \omega = \mathcal{O}(\varepsilon_j)$, $\int_{\beta_j} \omega = \mathcal{O}(\varepsilon_j)$, and $\int_{\partial \Gamma_j} \omega = \int_{\Gamma_j} d\omega = \mathcal{O}(\varepsilon_j)$, since $|D\Gamma_j| = \mathcal{O}(1)$. □

Prop. 16e10 is basically the converse to Prop. 11e11 for $k = 1$, and generalizes readily to all k.

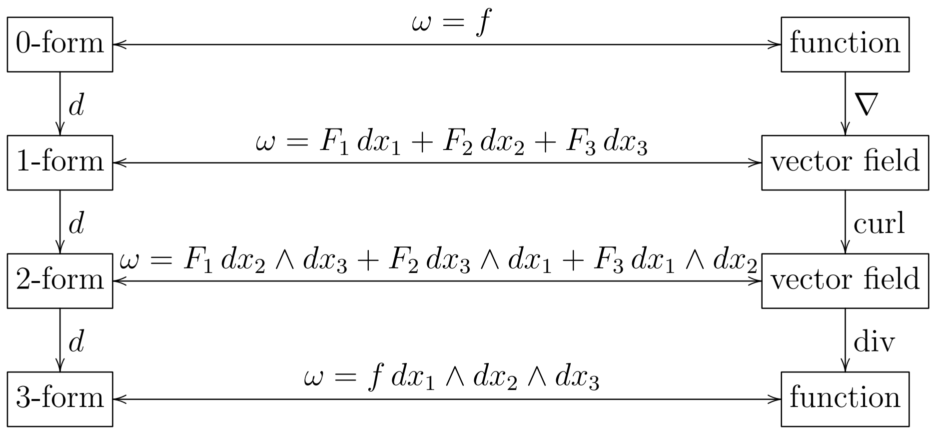
ON VECTOR CALCULUS

We know (recall Sect. 11e) that 0-forms and n-forms on $\mathbb{R}^n$ correspond to scalar fields (that is, functions), and no wonder: $\binom{n}{0} = \binom{n}{n} = 1$. Further, we know (recall (15d1)) that $(n-1)$-forms correspond to vector fields. Also 1-forms correspond to vector fields,

$$F_1 \, dx_1 + \dots + F_n \, dx_n \longleftrightarrow F \, ,$$

and no wonder: $\binom{n}{1} = \binom{n}{n-1} = n$. For other k it is harder to visualize k-forms, since $\binom{n}{k} > n$.

Dimension 3 is of special interest, and luckily, for $n = 3$ the four cases $0, 1, (n-1), n$ exhaust all k. A single notion "exterior derivative" corresponds (for $n = 3$) to three well-known operations of vector calculus: gradient (∇), curl (curl), and divergence (div), as follows.
(16e12)



Using 16b9, 16b10, $d(F_1\,dx_1 + F_2\,dx_2 + F_3\,dx_3) = dF_1 \wedge dx_1 + dF_2 \wedge dx_2 + dF_3 \wedge dx_3 = (D_1F_2 - D_2F_1)\,dx_1 \wedge dx_2 + (D_2F_3 - D_3F_2)\,dx_2 \wedge dx_3 + (D_3F_1 - D_1F_3)\,dx_3 \wedge dx_1$, thus,

(16e13) $\quad \operatorname{curl}(F_1, F_2, F_3) = (D_2F_3 - D_3F_2, D_3F_1 - D_1F_3, D_1F_2 - D_2F_1)\,.$

Stokes' theorem for $k = 2$, $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$ for a 1-form ω on $\mathbb{R}^3$, gives the *"classical Stokes' theorem"* (also known as "Kelvin-Stokes theorem", "curl theorem" and "Stokes' formula"): for every[1] vector field F (of class C^1) on $\mathbb{R}^3$ and every singular 2-box Γ in $\mathbb{R}^3$,

(16e14) the circulation of F around $\gamma = \partial\Gamma$

is equal to the flux of $\operatorname{curl} F$ through Γ ,

the circulation of F around γ being defined as $\int_{t_0}^{t_1} \langle F(\gamma(t)), \gamma'(t) \rangle \,\mathrm{d}t$.

In this sense, the curl is the circulation density, called also "vorticity" (and its flux is called also the net vorticity of F throughout Γ). A small paddle-wheel in the flow spins the fastest when its axle points in the direction of the curl vector, and in this case its angular speed is half the length of the curl vector.[2]

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[1] Since every F corresponds to some ω.
[2] Shifrin p. 394.