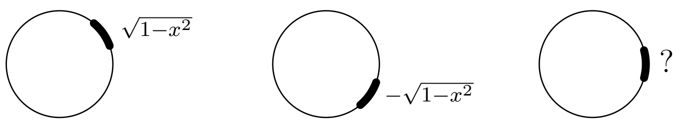
5 Implicit function theorem

Having more variables than equations we can express some variables in terms of the others, but only locally. The inverse function theorem helps a lot.

5a What is the problem

The set $Z = Z_g = \{(x, y) : g(x, y) = 0\} \subset \mathbb{R}^2$ for a given $g : \mathbb{R}^2 \to \mathbb{R}$ was discussed in Sect. 3, starting on Sect. 3a. Assuming that g is continuously differentiable near a given point $(x_0, y_0) \in Z$ and $\nabla g(x_0, y_0) \neq 0$ we got in Sect. 3d a path within Z through (x_0, y_0). In spite of Remark 3d3(a) this path is continuously differentiable, see Remark 4c12.

But why just a path? It would be nicer to have the graph of a function $\varphi : \mathbb{R} \to \mathbb{R}$. This is not always possible. For example, the circle $\{(x, y) : x^2 + y^2 = 1\} = Z_g$ for $g(x, y) = x^2 + y^2 - 1$. It is locally the graph of one of two functions $x \mapsto \sqrt{1 - x^2}$, $x \mapsto -\sqrt{1 - x^2}$ near $(x_0, y_0) \in Z_g$ except for $(x_0, y_0) = (\pm 1, 0)$. [Sh:p.206]



For this reason we assume that the vector $\nabla g(x_0, y_0)$ is not horizontal, that is,

$$(D_2 g)_{(x_0,y_0)} \neq 0 \, .$$

(Otherwise $x = \psi(y)$ rather than $y = \varphi(x)$.)

Hopefully, the section $g_x : \mathbb{R} \to \mathbb{R}$ of g defined by $g_x(y) = g(x, y)$ can help; it satisfies $g'_x \neq 0$ near y_0 (provided that x is close enough to x_0), which should imply existence and uniqueness of a root $y = \varphi(x)$ of g_x close to y_0. By some effort we could prove continuity of φ and maybe, by more effort, its continuous differentiability.

Of course, we need a multidimensional theory; $\mathbb{R}^2$ is only the simplest case.

Fortunately we do not need the effort mentioned above, since we have a powerful helper, — the inverse function theorem!

5b Simple observations before the theorem

Dimension $1 + 1 = 2$ (planar curves)

In Sect. 3d we got a path γ parametrized by values of the objective function f. Now we need the graph of a function φ; such graph may be thought of as a path parametrized by the coordinate x. Similarly to Sect. 3d we define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x,y) = \big(f(x,y), g(x,y)\big)$, but this time for $f(x,y) = x$. That is, we define

$$h(x,y) = \big(x, g(x,y)\big) .$$

We have

$$(Dh)_{(x_0,y_0)} = T = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$$

where $a = (D_1 g)_{(x_0,y_0)}$, $b = (D_2 g)_{(x_0,y_0)}$. Here and henceforth we ignore the distinction between operators and their matrices, and write vectors as columns (when matrices are involved). That is,

$$(Dh)_{(x_0,y_0)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ au + bv \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} ;$$

$$h(x_0 + u, y_0 + v) - h(x_0, y_0) = \begin{pmatrix} x_0 + u \\ g(x_0 + u, y_0 + v) \end{pmatrix} - \begin{pmatrix} x_0 \\ g(x_0, y_0) \end{pmatrix} = \begin{pmatrix} u \\ au + bv \end{pmatrix} + o(|\cdot|) .$$

Claim: T is invertible if and only if $b \ne 0$. Proof: $\det(T) = b$. Another proof:

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \Longleftrightarrow \quad (u = \xi, au + bv = \eta) \quad \Longleftrightarrow \quad (u = \xi, bv = \eta - a\xi) .$$

Dimension $3 + 2 = 5$ (for example)

We consider two constraints on 5 variables:

$$\begin{cases} g_1(x_1, x_2, x_3, y_1, y_2) = 0\,, \\ g_2(x_1, x_2, x_3, y_1, y_2) = 0\,, \end{cases}$$

that is,

$$g(x, y) = 0\,, \quad \text{where} \quad \begin{gathered} x = (x_1, x_2, x_3),\ y = (y_1, y_2), \\ g(x, y) = (g_1(x, y), g_2(x, y)); \end{gathered}$$

$g : \mathbb{R}^5 \to \mathbb{R}^2$.[1] We define $h : \mathbb{R}^5 \to \mathbb{R}^5$ by

$$h(x, y) = \begin{pmatrix} x \\ g(x, y) \end{pmatrix} \quad \text{for } x \in \mathbb{R}^3, y \in \mathbb{R}^2$$

and differentiate it at a given point $(x_0, y_0) = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, y_1^{(0)}, y_2^{(0)}) \in \mathbb{R}^5$:

$$(Dh)_{(x_0,y_0)} = T = \left(\begin{array}{c|c} \mathrm{id} & 0 \\ \hline A & B \end{array}\right); \tag{5b1}$$

here $\mathrm{id} \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ is the identity operator, and $(\,A \big| B\,) = (Dg)_{(x_0,y_0)} \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^2)$, $A \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$, $B \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$. Namely,

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{pmatrix},$$

the partial derivatives being taken at (x_0, y_0). A shorter notation:

$$A = \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}, \quad B = \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)}.$$

Thus,

$$\begin{aligned} g(x_0 + u, y_0 + v) - g(x_0, y_0) &= Au + Bv + o(|\cdot|) \\ h(x_0 + u, y_0 + v) - h(x_0, y_0) &= \begin{pmatrix} u \\ Au + Bv \end{pmatrix} + o(|\cdot|) \end{aligned} \quad \text{for } u \in \mathbb{R}^3, v \in \mathbb{R}^2.$$

Claim: T is invertible if and only if B is invertible. Proof: $\det(T) = \det(B)$. Another proof:

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \iff \quad (u = \xi, Bv = \eta - A\xi)\,. \tag{5b2}$$

[1] As before, we ignore the distinction between $\mathbb{R}^3 \times \mathbb{R}^2$ and $\mathbb{R}^5$.

Dimension $r + c = n$ (the general case)

This is completely similar to the case $r = 3, c = 2, n = 5$. Here $r, c \in \{1, 2, 3, \dots\}$; c constraints on n variables are given; $g : \mathbb{R}^r \times \mathbb{R}^c = \mathbb{R}^n \to \mathbb{R}^c$; $x_0 \in \mathbb{R}^r$, $y_0 \in \mathbb{R}^c$; $h : \mathbb{R}^n \to \mathbb{R}^n$;

$$A = \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}, \quad B = \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)};$$

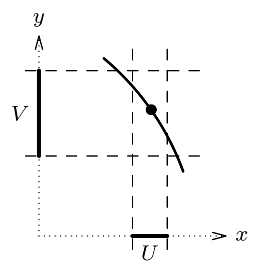
r may be called the number of degrees of freedom; 5b1 and 5b2 still apply.

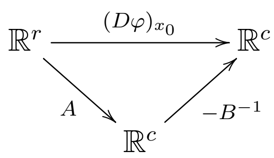
5c The theorem

5c1 Theorem. [Sh:Th.5.3.2] Assume that $r, c \in \{1, 2, 3, \dots\}$, $n = r + c$, $x_0 \in \mathbb{R}^r$, $y_0 \in \mathbb{R}^c$, $g : \mathbb{R}^n \to \mathbb{R}^c$ is continuously differentiable near (x_0, y_0), $g(x_0, y_0) = 0$, and the operator $B = \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)}$ is invertible. Then there exist open neighborhoods U of x_0 and V of y_0 such that

(a) for every $x \in U$ there exists one and only one $y \in V$ satisfying $g(x, y) = 0$;

(b) a function $\varphi : U \to V$ defined by $g\big(x, \varphi(x)\big) = 0$ is continuously differentiable, and $(D\varphi)_{x_0} = -B^{-1}A$ where $A = \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}$.

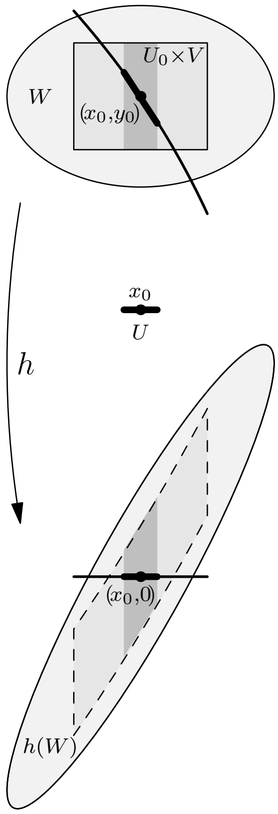




5c2 Remark. (a) The neighborhoods U, V can be chosen to be open balls (see the proof);

(b) the graph of φ covers *all* points of Z_g within $U \times V$ (recall 3d3(b) and 4c12).

Proof *of Theorem 5c1.*



A mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ g(x,y) \end{pmatrix} \quad \text{for } x \in \mathbb{R}^r, y \in \mathbb{R}^c$$

is continuously differentiable near (x_0, y_0), $h(x_0, y_0) = (x_0, 0)$, and the operator

$$(Dh)_{(x_0,y_0)} = T = \left(\begin{array}{c|c} \text{id} & 0 \\ \hline A & B \end{array}\right),$$

$$\text{where } A = \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}, \quad B = \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)},$$

is invertible (as shown in Sect. 5b). Theorem 4c2 gives an open neighborhood $W \subset \mathbb{R}^n$ of (x_0, y_0) such that $h(W) \subset \mathbb{R}^n$ is an open neighborhood of $(x_0, 0)$ and $h|_W$ is a diffeomorphism $W \to h(W)$.
We take open neighborhoods $U_0 \subset \mathbb{R}^r$ of x_0 and $V \subset \mathbb{R}^c$ of y_0 such that $U_0 \times V \subset W$. Using continuity of $(h|_W)^{-1}$ at $(x_0, 0)$ we take an open neighborhood $U \subset U_0$ of x_0 such that $U \times \{0\} \subset h(U_0 \times V)$.

Item (a), existence. Let $x \in U$, then $(h|_{U_0 \times V})^{-1}(x, 0) = (x, y)$ for some $y \in V$. We have $h(x, y) = (x, 0)$, therefore $g(x, y) = 0$.

Item (a), uniqueness. Let $x \in U$, $y_1, y_2 \in V$ and $g(x, y_1) = g(x, y_2) = 0$, then $h(x, y_1) = h(x, y_2) = (x, 0)$, therefore $y_1 = y_2$ (since h is one-to-one on W).

Item (b), continuous differentiability. The mapping φ satisfies (and may be defined by) the equality $\binom{x}{\varphi(x)} = (h|_W)^{-1}\binom{x}{0}$ for $x \in U$ (since $g(x, \varphi(x)) = 0$ implies $h(x, \varphi(x)) = (x, 0) \in h(W)$). The mapping $(h|_W)^{-1}$ is continuously differentiable on $h(W) \supset U \times \{0\}$. It follows that φ is continuously differentiable on U, since φ is the composition of three maps (two of them being linear): $x \mapsto \binom{x}{0}$; $(h|_W)^{-1}$; $\binom{x}{y} \mapsto y$. It remains to prove the formula for $(D\varphi)_{(x_0,y_0)}$.

The chain rule 2b11 in combination with 2b14 gives $(D\varphi)_{x_0}$ as the product of three operators: $x \mapsto \binom{x}{0}$; $\big(D(h|_W)^{-1}\big)_{(x_0,0)}$; $\binom{x}{y} \mapsto y$. By 4c4, $\big(D(h|_W)^{-1}\big)_{(x_0,0)} = \big((Dh)_{(x_0,y_0)}\big)^{-1} = T^{-1}$. We note that

$$T^{-1} = \left(\begin{array}{c|c} \text{id} & 0 \\ \hline -B^{-1}A & B^{-1} \end{array}\right)$$

since

$$T\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\xi\\\eta\end{pmatrix}\iff\begin{Bmatrix}u=\xi\\Bv=\eta-A\xi\end{Bmatrix}\iff\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\xi\\B^{-1}(\eta-A\xi)\end{pmatrix}.$$

It follows that

$$T^{-1}\begin{pmatrix}\xi\\0\end{pmatrix}=\begin{pmatrix}\xi\\-B^{-1}A\xi\end{pmatrix},$$

and finally, $(D\varphi)_{x_0}(\xi)=-B^{-1}A\xi$. □

If in the linear approximation y is a function of x then y is locally a function of x.

5c3 Exercise. In dimension $1+1=2$ prove that

$$\varphi'(x_0)=-\frac{(D_1g)_{(x_0,y_0)}}{(D_2g)_{(x_0,y_0)}}.$$

(Less formally, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\partial g/\partial x}{\partial g/\partial y}$ since $\frac{\partial g}{\partial x}\mathrm{d}x+\frac{\partial g}{\partial y}\mathrm{d}y=\mathrm{d}g(x,y)=0$.)

5c4 Exercise. (a) In dimension $1+1=2$, assuming in addition that $g\in C^2$ near (x_0,y_0), prove that $\varphi\in C^2$ near x_0.[1]

(b) Generalize (a) to arbitrary dimension $r+c=n$ and C^k for arbitrary $k=1,2,3,\dots$

5c5 Exercise. (a) Prove that the equation $x\cos xy=0$ near the point $(x,y)=(1,\pi/2)$ has unique solution $y=y(x)$, and the function $y(\cdot)$ is convex near $x=1$.

(b) The same for the equation $xy+\ln x+\ln y=1$ near the point $(x,y)=(1,1)$.[2]

5c6 Exercise. Given $k\in\{1,2,3,\dots\}$, we define $g:\mathbb{R}^2\to\mathbb{R}$ by $g(x,y)=\operatorname{Im}\big((x+\mathrm{i}y)^k\big)$. Also, let $x_0=y_0=0$.

(a) Find all k such that $g(\cdot,\cdot),x_0,y_0$ satisfy the *assumptions* of Theorem 5c1.

(b) Find all k such that $g(\cdot,\cdot),x_0,y_0$ satisfy the *conclusions* of Theorem 5c1.

5c7 Exercise. Let $g(\cdot,\cdot),x_0,y_0$ satisfy the assumptions of Theorem 5c1 for $c=1$. Show that $g^2(\cdot,\cdot),x_0,y_0$ violate the assumptions of Theorem 5c1 but still satisfy its conclusions.

---

[1] Hint: $\varphi'(x)=-\frac{(D_1g)_{(x,\varphi(x))}}{(D_2g)_{(x,\varphi(x))}}$.

[2] Hint: in both cases you can do with almost no calculations!

5c8 Exercise. Assume that $g : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable near the origin, and $(D_1 g)_0 \neq 0, \dots, (D_n g)_0 \neq 0$. Then the equation $g(x_1, \dots, x_n) = 0$ locally defines n functions $x_1(x_2, \dots, x_n)$, $x_2(x_1, x_3, \dots, x_n)$, $\dots$, $x_n(x_1, \dots, x_{n-1})$. Find the product

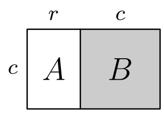
$$\frac{\partial x_1}{\partial x_2} \frac{\partial x_2}{\partial x_3} \dots \frac{\partial x_{n-1}}{\partial x_n} \frac{\partial x_n}{\partial x_1}$$

at the origin.[1]

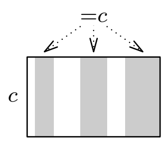
5d Degrees of freedom

In Theorem 3f1 linear independence of $\nabla g_1(x_0), \dots, \nabla g_m(x_0)$ is required. In Theorem 5c1 the operator $B = \frac{\partial g}{\partial y}\big|_{(x_0,y_0)}$ is required to be invertible. Is it the same, or not?

The number of constraints, denoted by m in Sect. 3, is now denoted by c. The number of variables is n (in both cases).

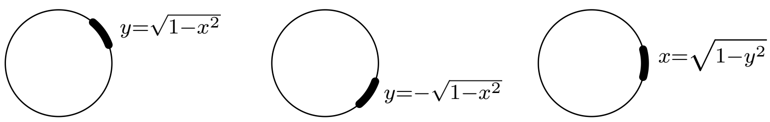


The matrix of B is a part of the matrix $(A \,|\, B) = (Dg)_{(x_0,y_0)}$.
The vectors $\nabla g_1(x_0), \dots, \nabla g_c(x_0)$ are the rows of the matrix $(A \,|\, B) = (Dg)_{(x_0,y_0)}$.
If they are linearly dependent then clearly the rows of B are, and therefore B is not invertible. The contrary is wrong: it can happen that the rows of B are linearly dependent but the rows of $(A \,|\, B)$ are not.



Linear independence of the rows of $(A \,|\, B)$ means that this matrix is of rank c, that is, some $c \times c$ minor is not zero. But not just the rightmost minor. That is, some c out of the n variables are functions of the other r variables. But not just the last c variables of the first r variables.

Recall the circle treated in Sect. 5a:



Generally, dealing with a set of the form

$$Z_g = \{x : g(x) = 0\}, \quad g : \mathbb{R}^n \to \mathbb{R}^c$$

and a point $x_0 \in Z_g$ such that $\nabla g_1(x_0), \dots, \nabla g_c(x_0)$ are linearly independent (equivalently, the operator $(Dg)_{x_0}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^c$, recall 2f2), we see that near x_0 the set Z_g is basically the graph of a continuously differentiable

[1] Hint: first, consider a linear g.

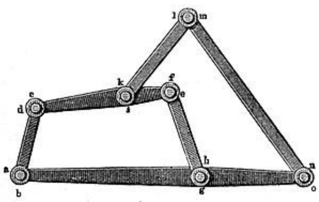
mapping $\mathbb{R}^r \to \mathbb{R}^c$ (but the division of n variables into r and c may depend on x_0). In this situation one says that Z_g at x_0 has r degrees of freedom.[1]

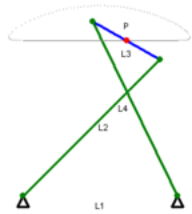
In particular, the circle has one degree of freedom at every point.

5e Examples

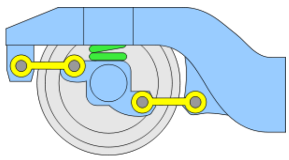
Planar linkages[2]

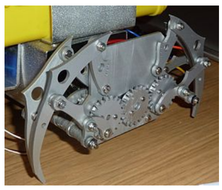
Informally, planar linkages are mechanical systems like these:[3]





They are used in machines and tools.[4]





They consist of bodies, called bars or links, connected by joints.

The number of degrees of freedom is important. One degree of freedom is the main case in classical mechanics. Zero degrees of freedom means that the system is rather a construction. Several degrees of freedom are widely used in robotics.

Some planar links may be described as finite sequences $(A_1, \dots, A_p)$ of points of $\mathbb{R}^2$ constrained by equations of two forms, $|A_i - A_j| = c_{i,j}$ and $|A_i - B_k| = c'_{i,k}$; here $c_{i,j}$ and $c'_{i,k}$ are lengths of the links and B_k are fixed

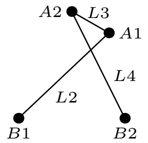
[1]This is basically the dimension of a manifold, treated in Analysis 4.

[2]"Linkages of rods are everywhere, in mechanics (consider a railway bridge or the Eiffel tower), in biology (the skeleton), in robotics, in chemistry." Hubbard, Example 3.1.8, p. 297.

[3]Images from Wikipedia (articles "Six-bar linkage", "Chebyshev linkage").

[4]Images from Wikipedia (articles "Four-bar linkage", "Klann linkage", "Linkage (mechanical)").

points. An example:



$$|A_1 - B_1| = L_2\,,$$
$$|A_1 - A_2| = L_3\,,$$
$$|A_2 - B_2| = L_4\,.$$

Here we observe 2 fixed points, $p = 2$ free points and $c = 3$ constraints, and expect one degree of freedom ($2p - c = 1$).

Generally, given p (free) points and c constraints, one expects to have

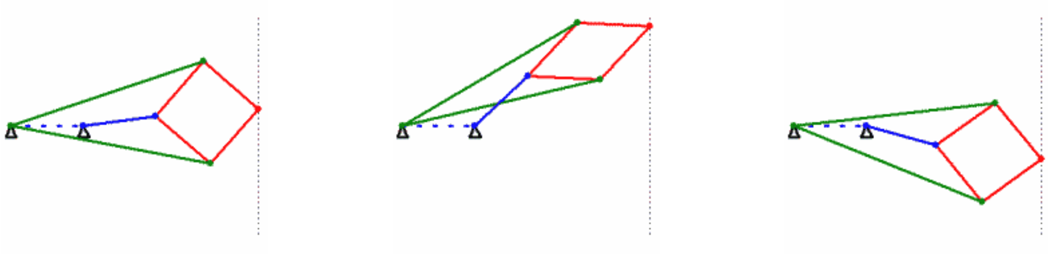
$$M = 2p - c$$

degrees of freedom. This M is called mobility. Is it really the number of degrees of freedom?

The functions $|A_i - A_j|$ are continuously differentiable everywhere except for the case $A_i = A_j$. We could turn to the functions $|A_i - A_j|^2$ continuously differentiable everywhere, but the case $A_i = A_j$ remains exceptional: the gradient vanishes here. Well, we assume that $c_{i,j} > 0$ and $c'_{i,k} > 0$, which ensures continuous differentiability and non-zero gradients. The question is, are the gradients linearly independent?

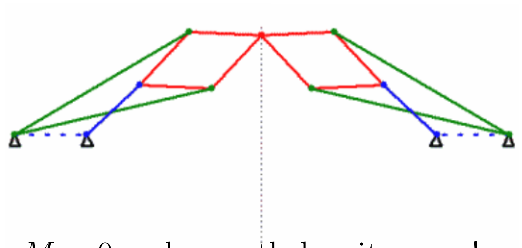
The answer appears to be affirmative for majority of practically important cases. But not always. Sometimes the answer is negative; such planar linkages are called *overconstrained*. Putting aside trivial examples such as $L_{1,3} = L_{1,2} + L_{2,3}$ we turn to a quite nontrivial example.

The so-called Peaucellier-Lipkin linkage transforms rotation into straight motion.[1]



[1]Images from Wikipedia (article "Peaucellier-Lipkin linkage").

Here $p = 4$, $c = 7$, $M = 1$. We "symmetrize" this linkage as follows:



Now $p = 7$, $c = 14$, $M = 0$ and nevertheless it moves!

Returning to the Peaucellier-Lipkin linkage we denote by $f(\cdot)$ the horizontal coordinate of the rightmost point, observe that $f(\cdot) = \text{const}$ on Z_g, thus, f has a local constrained extremum (not strict, of course) at every point of Z_g. Taking for granted that this linkage is not overconstrained we conclude that $\nabla f(x) = \lambda_1 \nabla g_1(x) + \cdots + \lambda_7 \nabla g_7(x)$ for some $\lambda_1, \dots, \lambda_7$ (depending on $x \in Z_g$).

For the symmetrized linkage we still have $\nabla f(x) = \lambda_1 \nabla g_1(x) + \cdots + \lambda_7 \nabla g_7(x)$, but also, by symmetry, $-\nabla f(x) = \lambda_1 \nabla g_8(x) + \cdots + \lambda_7 \nabla g_{14}(x)$. Clearly, $\lambda_1, \dots, \lambda_7$ are not all zero, and $\lambda_1 \nabla g_1(x) + \cdots + \lambda_7 \nabla g_7(x) + \lambda_1 \nabla g_8(x) + \cdots + \lambda_7 \nabla g_{14}(x) = 0$ is a linear dependence between the gradients of the constraints.

Rotations in three dimensions

A rotation of $\mathbb{R}^3$ is a linear operator $U : \mathbb{R}^3 \to \mathbb{R}^3$ such that $|Ux| = |x|$ for all $x \in \mathbb{R}^3$ and in addition $\det U = 1$ (rather than -1); here U is thought of as a 3×3 matrix. The three columns u_1, u_2, u_3 of this matrix $U = (u_1 \,|\, u_2 \,|\, u_3)$ are orthonormal vectors in $\mathbb{R}^3$. In addition, $u_3 = u_1 \times u_2$ (the cross product, called also vector product). We may describe U by a pair (u_1, u_2) of orthonormal vectors (the correspondence $U \leftrightarrow (u_1, u_2)$ being bijective). Now the situation is similar to that of "planar linkages", but spatial rather than planar. We have one fixed point $O = (0, 0, 0) \in \mathbb{R}^3$, two free points $u_1, u_2 \in \mathbb{R}^3$ and three constraints

$$|u_1 - O| = 1\,, \quad |u_2 - O| = 1\,, \quad |u_1 - u_2| = \sqrt{2} \quad \text{equivalent to}$$
$$|u_1| = 1\,, \quad |u_2| = 1\,, \quad \langle u_1, u_2 \rangle = 0\,.$$

Thus, $p = 2$; the number of variables is $3p = 6$; the number of constraints is $c = 3$; the mobility is $M = 3p - c = 3$. Is it really the number of degrees of freedom? That is, are the gradients (of the constraints) linearly independent?

We take $u_1 = (x_1, x_2, x_3)$, $u_2 = (x_4, x_5, x_6)$, and

$$g_1(x_1, \dots, x_6) = x_1^2 + x_2^2 + x_3^2 - 1\,,$$
$$g_2(x_1, \dots, x_6) = x_4^2 + x_5^2 + x_6^2 - 1\,,$$
$$g_3(x_1, \dots, x_6) = x_1 x_4 + x_2 x_5 + x_3 x_6\,.$$

Thus,

$$\nabla g_1(x_1,\dots,x_6) = (2x_1, 2x_2, 2x_3, 0, 0, 0)\,,$$
$$\nabla g_2(x_1,\dots,x_6) = (0, 0, 0, 2x_4, 2x_5, 2x_6)\,,$$
$$\nabla g_3(x_1,\dots,x_6) = (x_4, x_5, x_6, x_1, x_2, x_3)\,.$$

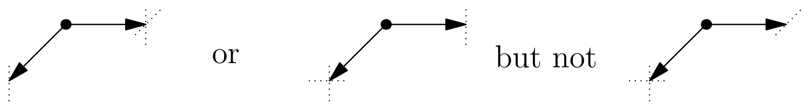
If $\lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \lambda_3 \nabla g_3 = 0$ at a point of Z_g then $2\lambda_1 u_1 + \lambda_3 u_2 = 0$ and $2\lambda_2 u_2 + \lambda_3 u_1 = 0$ (check it), which implies $\lambda_1 = \lambda_2 = \lambda_3 = 0$, since $|2\lambda_1 u_1 + \lambda_3 u_2|^2 = 4\lambda_1^2 + \lambda_3^2$ and $|2\lambda_2 u_2 + \lambda_3 u_1|^2 = 4\lambda_2^2 + \lambda_3^2$.

We see that the three gradients are linearly independent, and therefore Z_g has 3 degrees of freedom at every point.

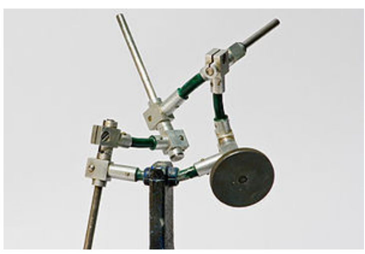
For instance, at the point $u_1 = (1, 0, 0), u_2 = (0, 1, 0)$ we have

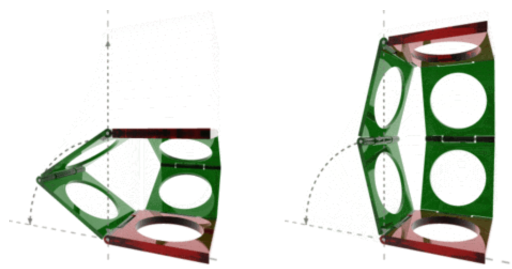
$$\begin{pmatrix} \nabla g_1 \\ \hline \nabla g_2 \\ \hline \nabla g_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}\;;$$

a non-zero 3×3 minor can be chosen in two ways: by taking columns $1, 2, 5$ or $1, 4, 5$. It means that near the given point x_1, x_2, x_5 are functions of x_3, x_4, x_6; or alternatively,x_1, x_4, x_5 are functions of x_2, x_3, x_6.



Now we could turn to spatial (rather than planar) linkages, in particular, the overconstrained Sarrus linkage.[1]





However, this is beyond our course.

[1]Images from Wikipedia (articles "Four-bar linkage", "Sarrus linkage").