7 Iterated integral

Iterated integral is an indispensable tool for calculating multidimensional integrals (in particular, volumes). It also leads to a result about integrals (including one-dimensional) that depend on a parameter.

7a What is the problem

It is easy to see that

$$\varepsilon^2 \sum_{k,l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) \to \int_{\mathbb{R}^2} f \quad \text{as } \varepsilon \to 0$$

for every continuous $f : \mathbb{R}^2 \to \mathbb{R}$ with bounded support. The double summation is evidently equivalent to iterated summation,

$$\varepsilon^2 \sum_{k,l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) = \varepsilon \sum_{k\in\mathbb{Z}} \biggl(\varepsilon \sum_{l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) \biggr) \,,$$

which suggests that

$$\int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \biggl(\int_{\mathbb{R}} f(x,y) \, \mathrm{d}y \biggr) \mathrm{d}x \,,$$

(alternative notation: $\iint f(x,y) \, \mathrm{d}x\mathrm{d}y = \int \mathrm{d}x \int \mathrm{d}y \, f(x,y)$, and the like), that is,

(7a1)
$$\int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \biggl(x \mapsto \int_{\mathbb{R}} f(x,\cdot) \biggr) \,,$$

where $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ (denoted also f_x) is defined by

$$f(x,\cdot) : y \mapsto f(x,y) \,.$$

It should be very useful, to integrate with respect to one variable at a time.

Related problems:

* $*$ does integrability of f imply integrability of $f(x,\cdot)$ for every x?
* $*$ is the function $x \mapsto \int_{\mathbb{R}} f(x,\cdot)$ integrable?
* $*$ is the two-dimensional integral equal to the iterated integral?
* $*$ if the iterated integral is well-defined, does it follow that f is integrable?

And, of course, we need a multidimensional theory; $\mathbb{R}^2$ is only the simplest case.

7b Lipschitz functions

Here is the so-called *Lipschitz condition* (with constant L) on a function $f : \mathbb{R}^n \to \mathbb{R}$:

(7b1) $$|f(x) - f(y)| \le L|x-y| \quad \text{for all } x, y\,.$$

One also says that f is Lipschitz continuous (with constant L), or L-Lipschitz, etc. Also, f is Lipschitz continuous if it satisfies the Lipschitz condition with some constant. Such functions are continuous (but the converse fails). The same holds for functions on boxes and other subsets of $\mathbb{R}^n$.

Every Lipschitz function on a box is uniformly continuous and therefore integrable by (6f1).

Every integrable function on a box can be sandwiched between Lipschitz functions (see 6f4 and (6f6); max and min of Lipschitz functions are Lipschitz functions, see the hint to 6f4).

7b2 Proposition. Let $f : B \to \mathbb{R}$ be a Lipschitz function on a box $B = b_1 \times b_2 \subset \mathbb{R}^2$. Then
(a) for every $x \in b_1$ the function f_x is Lipschitz continuous on b_2;
(b) the function $x \mapsto \int_{b_2} f_x$ is Lipschitz continuous on b_1;

(c) $$\int_B f = \int_{b_1} \Big(x \mapsto \int_{b_2} f_x \Big)\,.$$

It is given that f is L-Lipschitz for some $L \in (0,\infty)$. We reduce the general case to the case $L = 1$ by turning to the function $\frac{1}{L} f$.

We reduce the general box B of the form $[s_1,t_1] \times [s_2,t_2]$ to a box of the form $[0,t_1] \times [0,t_2]$ by translation, according to 6c3. Further, we reduce it to the square $[0,1] \times [0,1]$ by rescaling, according to 6d17. That is, we introduce a Lipschitz function $g : [0,1] \times [0,1] \to \mathbb{R}$ by $g(x,y) = f(t_1 x, t_2 y)$; by 6d17,

$$t_1 t_2 \int_{[0,1]\times[0,1]} g = \int_B f\,.$$

We note that $g_x(y) = f_{t_1 x}(t_2 y)$; Lipschitz continuity of g_x implies Lipschitz continuity of $f_{t_1 x}$, and $t_2 \int_{[0,1]} g_x = \int_{[0,t_2]} f_{t_1 x}$ (by 6d17 again). Further, Lipschitz continuity of $x \mapsto \int_{[0,1]} g_x$ implies Lipschitz continuity of $x \mapsto \int_{[0,t_2]} f_x$, and

$$t_1 \int_{[0,1]} \Big(x \mapsto \int_{[0,t_2]} f_{t_1 x} \Big) = \int_{[0,t_1]} \Big(x \mapsto \int_{[0,t_2]} f_x \Big)$$

(by 6d17 once again), that is,

$$t_1 t_2 \int_{[0,1]} \Big(x \mapsto \int_{[0,1]} g_x \Big) = \int_{[0,t_1]} \Big(x \mapsto \int_{[0,t_2]} f_x \Big) .$$

Now the equality (7b2)(c) for g implies the same for f.

7b3 Lemma. For every 1-Lipschitz function $f : [0,1]^n \to \mathbb{R}$ and every $K = 1, 2, \dots$

$$\bigg| \frac{1}{K^n} \sum_{1 \le k_1,\dots,k_n \le K} f\Big(\frac{k_1 - 0.5}{K}, \dots, \frac{k_n - 0.5}{K} \Big) - \int_{[0,1]^n} f \bigg| \le \frac{\sqrt{n}}{2K} .$$

Proof. Let us define[1] a closed δ-pixel as a box (cube) of the form $[\delta k_1, \delta k_1 + \delta] \times \dots \times [\delta k_n, \delta k_n + \delta]$ for $k_1, \dots, k_n \in \mathbb{Z}$. We consider a partition P of $[0,1]^n$ into K^n δ-pixels with $\delta = 1/K$. Every point of a pixel C is $\frac12 \delta \sqrt{n}$-close to the center $(\frac{k_1 - 0.5}{K}, \dots, \frac{k_n - 0.5}{K})$ of the pixel; the Lipschitz continuity gives

$$f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) - \frac12 \delta \sqrt{n} \le \inf_C f \le \sup_C f \le f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) + \frac12 \delta \sqrt{n} ;$$

$$\sum_{1 \le k_1,\dots,k_n \le K} \delta^n \Big(f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) - \frac12 \delta \sqrt{n} \Big) \le L(f,P) \le \int_{[0,1]^n} f \le$$

$$\le U(f,P) \le \sum_{1 \le k_1,\dots,k_n \le K} \delta^n \Big(f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) + \frac12 \delta \sqrt{n} \Big) .$$

□

Proof *of Prop. 7b2 for a* 1*-Lipschitz function* f *on* $B = [0,1] \times [0,1]$.

(a) $|f_x(y_1) - f_x(y_2)| = |f(x,y_1) - f(x,y_2)| \le |(0, y_1 - y_2)| = |y_1 - y_2|$, thus f_x is 1-Lipschitz.

(b) $|(f_{x_1} - f_{x_2})(y)| = |f(x_1,y) - f(x_2,y)| \le |(x_1 - x_2, 0)| = |x_1 - x_2|$, therefore $| \int_{[0,1]} f_{x_1} - \int_{[0,1]} f_{x_2} | \le |x_1 - x_2|$, which shows that the function $x \mapsto \int_{[0,1]} f_x$ is 1-Lipschitz.

[1] Following Terence Tao.

(c) Lemma 7b3 applied to f (and $n=2$) gives for arbitrary $M = 1, 2, \dots$

$$\biggl| \frac{1}{M^2} \sum_{k,l=1}^M f\bigl(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M} \bigr) - \int_B f \biggr| \le \frac{\sqrt{2}}{2M} \, .$$

The same lemma applied to f_x (and $n=1$) gives for every x

$$\biggl| \frac{1}{M} \sum_{l=1}^M f_x\bigl(\tfrac{l-0.5}{M} \bigr) - \int_{[0,1]} f_x \biggr| \le \frac{1}{2M} \, .$$

The same lemma (again!) applied to the function $x \mapsto \int_{[0,1]} f_x$ (and $n=1$) gives

$$\biggl| \frac{1}{M} \sum_{k=1}^M \int_{[0,1]} f_{\frac{k-0.5}{M}} - \int_{[0,1]} \Bigl(x \mapsto \int_{[0,1]} f_x \Bigr) \biggr| \le \frac{1}{2M} \, .$$

Thus,

$$\biggl| \int_B f - \int_{[0,1]} \Bigl(x \mapsto \int_{[0,1]} f_x \Bigr) \biggr| \le \\ \le \biggl| \frac{1}{M^2} \sum_{k,l=1}^M f\bigl(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M} \bigr) - \frac{1}{M} \sum_{k=1}^M \int_{[0,1]} f_{\frac{k-0.5}{M}} \biggr| + \frac{\sqrt{2}}{2M} + \frac{1}{2M}$$

and

$$\biggl| \frac{1}{M^2} \sum_{k,l=1}^M f\bigl(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M} \bigr) - \frac{1}{M} \sum_{k=1}^M \int_{[0,1]} f_{\frac{k-0.5}{M}} \biggr| = \\ = \biggl| \frac{1}{M} \sum_{k=1}^M \biggl(\frac{1}{M} \sum_{l=1}^M f\bigl(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M} \bigr) - \int_{[0,1]} f_{\frac{k-0.5}{M}} \biggr| \le \\ \le \frac{1}{M} \sum_{k=1}^M \frac{1}{2M} = \frac{1}{2M} \, .$$

Finally,

$$\biggl| \int_B f - \int_{[0,1]} \Bigl(x \mapsto \int_{[0,1]} f_x \Bigr) \biggr| \le \frac{\sqrt{2}+2}{2M}$$

for all M. □

Here is a straightforward generalization of Prop. 7b2.

7b4 Proposition. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and a Lipschitz function f on a box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then
(a) for every $x \in B_1$ the function f_x is Lipschitz continuous on B_2;
(b) the function $x \mapsto \int_{B_2} f_x$ is Lipschitz continuous on B_1;

(c)
$$\int_B f = \int_{B_1} \Big(x \mapsto \int_{B_2} f_x \Big) \,.$$

7b5 Exercise. Prove Prop. 7b4.

Similarly, for a Lipschitz function $f : B_1 \times B_2 \to \mathbb{R}$,

$$\int_B f = \int_{B_2} \Big(y \mapsto \int_{B_1} f^y \Big)$$

where $f^y(x) = f(x,y)$. This claim reduces to Prop. 7b4 taking $\tilde{f}(y,x) = f(x,y)$. Ultimately,

$$\int \mathrm{d}x \int \mathrm{d}y\, f(x,y) = \iint f(x,y)\,\mathrm{d}x\mathrm{d}y = \int \mathrm{d}y \int \mathrm{d}x\, f(x,y) \,.$$

That is, the two iterated integrals are equal to the "non-iterated" ("double"? "single"?) integral (and therefore equal to each other).

7b6 Exercise. Prove that

$$\int_{B_1 \times B_2} f(x_1,\dots,x_m) g(y_1,\dots,y_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_m\,\mathrm{d}y_1 \dots \mathrm{d}y_n = \\ = \Big(\int_{B_1} f(x_1,\dots,x_m)\,\mathrm{d}x_1 \dots \mathrm{d}x_m \Big) \Big(\int_{B_2} g(y_1,\dots,y_n)\,\mathrm{d}y_1 \dots \mathrm{d}y_n \Big)$$

for Lipschitz functions $f : B_1 \to \mathbb{R}$, $g : B_2 \to \mathbb{R}$.

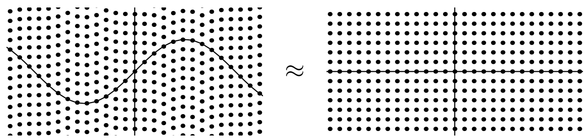
7b7 Exercise. Calculate each integral in two ways:
(a) $\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \mathrm{e}^{x+y}$;
(b) $\int_0^1 \mathrm{d}y \int_0^{\pi/2} \mathrm{d}x\, xy \cos(x+y)$.

7b8 Exercise. Calculate integrals
(a) $\int_{[0,1]^n} (x_1^2 + \dots + x_n^2)\,\mathrm{d}x_1 \dots \mathrm{d}x_n$;
(b) $\int_{[0,1]^n} (x_1 + \dots + x_n)^2\,\mathrm{d}x_1 \dots \mathrm{d}x_n$.

7b9 Exercise. For every Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ with bounded support,

$$\iint_{\mathbb{R}^2} f(x, y + \sin x) \, \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} f(x,y) \, \mathrm{d}x\mathrm{d}y \, .$$

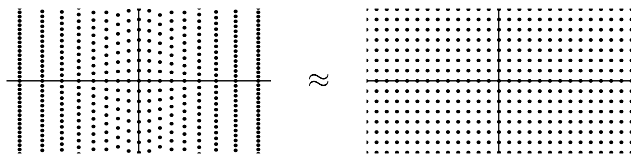
Prove it.



7b10 Exercise. For every Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ with bounded support,

$$\iint_{\mathbb{R}^2} f\Big(x^3 + x, \frac{y}{3x^2+1} \Big) \, \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} f(x,y) \, \mathrm{d}x\mathrm{d}y \, .$$

Prove it.



7c Some counterexamples

7c1 Example. Integrability of f does not imply integrability of f_x for every x.

Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x = 1/2 \text{ and } y \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(\cdot,\cdot) = 0$ outside a set $\{1/2\} \times [0,1]$ of area 0, therefore f is integrable (recall 6g). However, $f_{1/2}$ is not integrable (recall 6b29).

7c2 Example. Existence of the iterated integral[1] does not imply boundedness (the more so, integrability) of f, even if f is positive and symmetric in the sense that $f(x,y) = f(y,x)$ (and therefore the iterated integrals $\int \mathrm{d}x \int \mathrm{d}y \, f(x,y)$, $\int \mathrm{d}y \int \mathrm{d}x \, f(x,y)$ are both well-defined, and equal).

Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{x+y}} & \text{if } x/2 < y < 2x, \\ 0 & \text{otherwise} \end{cases}$$

[1]That is, integrability of f_x for all x *and* integrability of the function $x \mapsto \int f_x$.

and observe that

$$\int_{[0,1]} f_x = \int_{x/2}^{2x} \frac{\mathrm{d}y}{\sqrt{x+y}} = 2\sqrt{x+y}\Big|_{y=x/2}^{y=2x} = 2\sqrt{3x} - 2\sqrt{3x/2} = \text{const} \cdot \sqrt{x}$$

for $x \le 1/2$, and $\int_{x/2}^{1} \frac{\mathrm{d}y}{\sqrt{x+y}} = 2\sqrt{x+1} - 2\sqrt{3x/2}$ for $x \ge 1/2$.

7c3 Example. Existence of both iterated integrals does not imply their equality, even if f is antisymmetric in the sense that $f(x,y) = -f(y,x)$.

Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x-y}{(x+y)^3} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise;} \end{cases}$$

observe that each f_x is continuously differentiable (therefore Lipschitz), and

$$\int_{[0,1]} f_x = \int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y = \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,\mathrm{d}y =$$

$$= 2x \int_0^1 \frac{\mathrm{d}y}{(x+y)^3} - \int_0^1 \frac{\mathrm{d}y}{(x+y)^2} = 2x \cdot \Big(-\frac{1}{2}\Big) \frac{1}{(x+y)^2}\Big|_{y=0}^{y=1} - (-1) \cdot \frac{1}{x+y}\Big|_{y=0}^{y=1} =$$

$$= -x\Big(\frac{1}{(x+1)^2} - \frac{1}{x^2}\Big) + \Big(\frac{1}{x+1} - \frac{1}{x}\Big) = \frac{-x + (x+1)}{(x+1)^2} = \frac{1}{(x+1)^2},$$

a positive, continuously differentiable function on $[0,1]$. Its integral is positive (in fact, $1/2$). By the antisymmetry, the other iterated integral is negative (in fact, $-1/2$).

7c4 Example. Existence of the iterated integral does not imply integrability of f even if f is *bounded* and symmetric.

Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by[1]

$$f(x,y) = \begin{cases} 1 & \text{if } x\sqrt{2} + y \text{ and } x + y\sqrt{2} \text{ are (both) rational,} \\ 0 & \text{otherwise.} \end{cases}$$

If $f(x,y_1) = f(x,y_2) = 1$ then $y_1 - y_2 = (x\sqrt{2} + y_1) - (x\sqrt{2} + y_2)$ is rational and $(y_1 - y_2)\sqrt{2} = (x + y_1\sqrt{2}) - (x + y_2\sqrt{2})$ is rational, therefore $y_1 = y_2$. It means that each $f_x(\cdot) = 0$ outside at most one point. Similarly, each f^y vanishes outside at most one point. Thus, $\int f_x = 0$ for all x, and $\int f^y = 0$ for all y. Nevertheless f is not integrable, since it equals 1 on a dense countable set of points of the form $(q\sqrt{2} - r, r\sqrt{2} - q)$ with rational q, r; and f vanishes on the (dense) complement of this countable set.

[1] Alternatively, $f(x,y) = 1$ whenever $(x,y) = \big((2k-1)/2^n, (2l-1)/2^n\big)$.

Existence of an iterated integral does not ensure existence of the two-dimensional integral.

7c5 Remark. One may wonder, does existence of both iterated integrals imply their equality if f is just bounded (but not Lipschitz, nor integrable)? The answer is affirmative.[1] Try to prove it yourself if you are ambitious enough, but be warned that you'll probably need something not learned yet in this course.

7d Integrable functions

7d1 Theorem. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and an integrable function f on a box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then the iterated integrals

$$\int_{B_1} \mathrm{d}x \,{}_*\!\!\int_{B_2} \mathrm{d}y\, f(x,y)\,, \qquad \int_{B_1} \mathrm{d}x \,{}^*\!\!\int_{B_2} \mathrm{d}y\, f(x,y)\,,$$
$$\int_{B_2} \mathrm{d}y \,{}_*\!\!\int_{B_1} \mathrm{d}x\, f(x,y)\,, \qquad \int_{B_2} \mathrm{d}y \,{}^*\!\!\int_{B_1} \mathrm{d}x\, f(x,y)$$

are well-defined and equal to

$$\iint_B f(x,y)\,\mathrm{d}x\mathrm{d}y\,.$$

Clarification. The claim that $\int \mathrm{d}x \,{}_*\!\int \mathrm{d}y\, f(x,y)$ is well-defined means that the function $x \mapsto {}_*\!\int \mathrm{d}y\, f(x,y)$ is integrable.

The equality

$$\int \Big(x \mapsto {}_*\!\!\int f_x \Big) = \int \Big(x \mapsto {}^*\!\!\int f_x \Big)$$

implies integrability (with the same integral) of every function sandwiched between the lower and upper integrals. It is convenient to interpret $x \mapsto \int f_x$ as *any* such function and write, as before,

$$\int_B f = \int_{B_1} \Big(x \mapsto \int_{B_2} f_x \Big)$$

and

$$\int \mathrm{d}x \int \mathrm{d}y\, f(x,y) = \iint f(x,y)\,\mathrm{d}x\mathrm{d}y = \int \mathrm{d}y \int \mathrm{d}x\, f(x,y)$$

[1] In Riemann integration, of course. In Lebesgue integration the corresponding problem is much harder.

even though f_x may be non-integrable for some x.

Theorem 7d1 is proved via sandwiching (recall Sect. 6f), — either by step functions or by Lipschitz functions. Let us use the latter.

Proof. As was noted (before 7b2), ${}^*\!\!\int_B f = \inf_{g \ge f} \int_B g$ where g runs over all Lipschitz functions. For every such g, $\int_B g = \int_{B_1} (x \mapsto \int_{B_2} g_x)$ by Prop. 7b2. We have $\int_{B_2} g_x = {}^*\!\!\int_{B_2} g_x \ge {}^*\!\!\int_{B_2} f_x$ (since $g_x \ge f_x$), thus, $\int_B g \ge {}^*\!\!\int_{B_1} (x \mapsto {}^*\!\!\int_{B_2} f_x)$ for all these g. Therefore

$$ {}^*\!\!\int_B f \ge {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) . $$

Similarly (or via $(-f)$),

$$ {}_*\!\!\int_B f \le {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) . $$

Using integrability of f,

$$ \int_B f \le {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) \le {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) \le \int_B f \, , $$

therefore

$$ \int_B f = {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) = {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) . $$

Integrability of the function $x \mapsto {}_*\!\!\int_{B_2} f_x$ follows, since

$$ \int_B f = {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) \le {}^*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) \le {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) = \int_B f. $$

Similarly, the function $x \mapsto {}^*\!\!\int_{B_2} f_x$ is also integrable. Thus,

$$ \int_B f = \int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) = \int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) . $$

The other two iterated integrals are treated similarly (or via $\tilde f(y,x) = f(x,y)$). $\square$

7d2 Exercise. Give another proof of 7d1, via sandwiching by step functions.[1]

[1] Hint: first, consider $f = 1\!\!1_C$ for a box $C \subset B$.

7d3 Exercise. Generalize 7b6 to integrable functions

(a) assuming integrability of the function $(x, y) \mapsto f(x)g(y)$,

(b) deducing integrability of the function $(x, y) \mapsto f(x)g(y)$ from integrability of f and g (via sandwich).

7d4 Exercise. For every integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ the function $x, y \mapsto f(x, y + \sin x)$ is also integrable, and

$$\iint_{\mathbb{R}^2} f(x, y + \sin x) \, \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} f(x, y) \, \mathrm{d}x\mathrm{d}y \, .$$

Prove it.[1]

7d5 Exercise. For every integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ the function $x, y \mapsto f\big(x^3 + x, \frac{y}{3x^2+1}\big)$ is also integrable, and

$$\iint_{\mathbb{R}^2} f\Big(x^3 + x, \frac{y}{3x^2 + 1}\Big) \, \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} f(x, y) \, \mathrm{d}x\mathrm{d}y \, .$$

Prove it.[2]

7d6 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^n$ are Jordan measurable sets then the set $E = E_1 \times E_2 \subset \mathbb{R}^{m+n}$ is Jordan measurable.

Prove it.

Clearly, the boxes B_1, B_2 in Th. 7d1 may be replaced with Jordan sets E_1, E_2.

7d7 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^{m+n}$ are Jordan measurable sets then the set $E = \{(x, y) \in E_2 : x \in E_1\} = (E_1 \times \mathbb{R}^n) \cap E_2 \subset \mathbb{R}^{m+n}$ is Jordan measurable.

Prove it.

Applying Theorem 7d1 to a function $f 1\!\!1_E$ and taking 6g16 into account we get the following.

7d8 Corollary. Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be integrable on every box, and $E \subset \mathbb{R}^{m+n}$ a Jordan measurable set; then

$$\int_E f = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{E_x} f_x \Big)$$

where $E_x = \{y : (x, y) \in E\} \subset \mathbb{R}^n$ for $x \in \mathbb{R}^m$.

[1]Hint: use 7b9.

[2]Hint: use 7b10.

Clarification. First, note that $\{x : E_x \ne \emptyset\}$ is bounded, and $\int_\emptyset f_x = 0$. Second: it may happen that $\int_{E_x} f_x$ is ill-defined for some x; then it is interpreted as anything between ${}_*\!\int f_x 1\!\!1_{E_x}$ and ${}^*\!\int f_x 1\!\!1_{E_x}$.

In particular, taking $f(\cdot) = 1$ we get

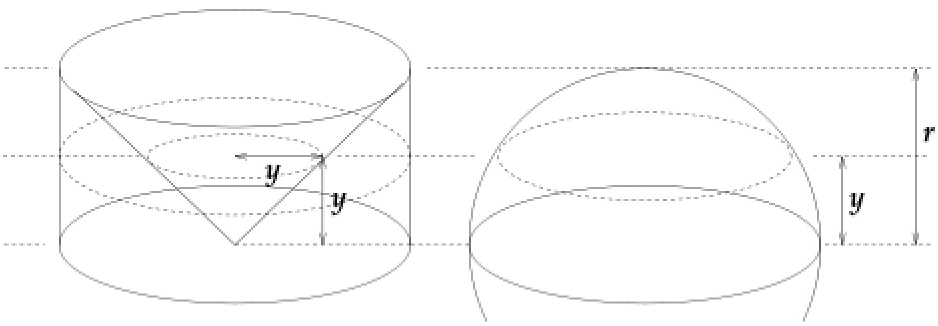
$$v_{m+n}(E) = \int_{\mathbb{R}^m} v_n(E_x)\,\mathrm{d}x \tag{7d9}$$

where v_k is the Jordan measure in $\mathbb{R}^k$. For instance, the volume of a 3-dimensional geometric body is the 1-dimensional integral of the area of the 2-dimensional section of the body.

7d10 Corollary. If Jordan measurable sets $E, F \subset \mathbb{R}^3$ satisfy $v_2(E_x) = v_2(F_x)$ for all x then $v_3(E) = v_3(F)$.[1]

This is a modern formulation of Cavalieri's principle:[2,3]
Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

Before emergence of the integral calculus, Cavalieri was able to calculate some volumes by ingenious use of this principle. Here are two examples. First, the volume of the upper half of a sphere is equal to the volume of a cylinder minus volume of a cone:



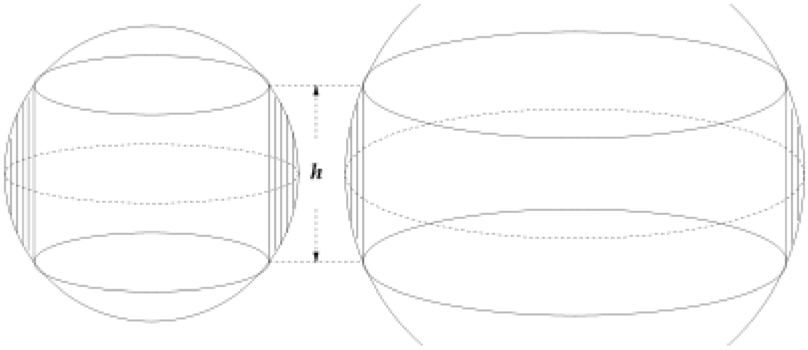
Second, when a hole of length h is drilled straight through the center of a sphere, the volume of the remaining material surprisingly does not depend

[1]It is sufficient to check the equality for all x of a dense subset of $\mathbb{R}$ (since two *Riemann integrable* functions equal on a dense set must have equal integrals).

[2]Bonaventura Francesco Cavalieri (in Latin, Cavalerius) (1598–1647), Italian mathematician.

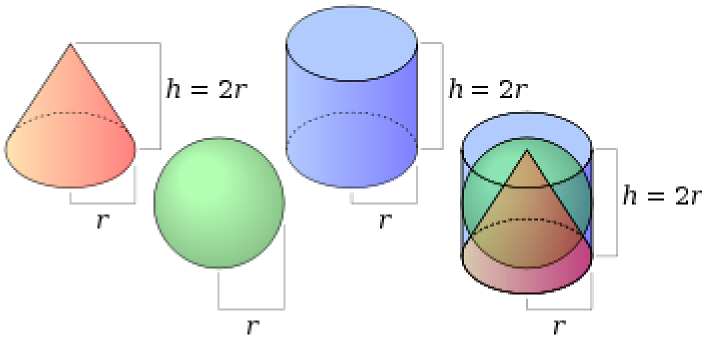
[3]Images (and some text) from Wikipedia, "Cavalieri's principle".

on the size of the sphere:



7d11 Exercise. Check the two results of Cavalieri noted above.

7d12 Exercise. Check a famous result of Archimedes:[1,2] a sphere inscribed within a cylinder has two thirds of the volume of the cylinder.



Moreover, show that the volumes of a cone, sphere and cylinder of the same radius and height are in the ratio 1 : 2 : 3.

Another important special case of 7d8:

$$E = \{(x,t) : x \in B, \, g(x) \le t \le h(x)\} \subset \mathbb{R}^{n+1}$$

where $B \subset \mathbb{R}^n$ is a box and $g.h : B \to \mathbb{R}$ integrable functions satisfying $g \le h$ (recall Sect. 6h). In this case $E_x = [g(x), h(x)]$, and we get

$$\int_E f = \int_B \Bigl(x \mapsto \int_{[g(x),h(x)]} f_x \Bigr) = \int_B \mathrm{d}x \int_{g(x)}^{h(x)} \mathrm{d}t \, f(x,t) \, .$$

---

[1]Archimedes ($\approx$ 287–212 BC), a Greek mathematician, generally considered to be the greatest mathematician of antiquity and one of the greatest of all time.
Cicero describes visiting the tomb of Archimedes, which was surmounted by a sphere inscribed within a cylinder. Archimedes ... regarded this as the greatest of his mathematical achievements.

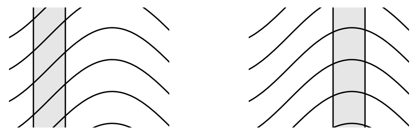
[2]Images (and some text) from Wikipedia, "Volume" (section "Volume ratios for a cone, sphere and cylinder of the same radius and height").

Applying this to $f 1\!\!1_F$ (in place of f) for a Jordan measurable set $F \subset \mathbb{R}^n$ such that $g \le h$ on F we get

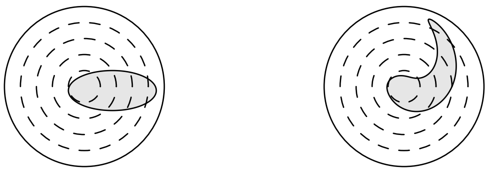
$$\int_E f = \int_F \mathrm{d}x \int_{g(x)}^{h(x)} \mathrm{d}t\, f(x,t)$$

where $E = \{(x,t) : x \in F, g(x) \le t \le h(x)\}$.

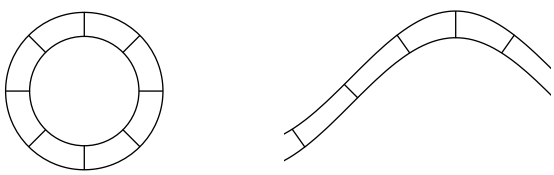
7d13 Remark. Cavalieri's principle is about parallel planes. What about parallel surfaces or curves? Applying 7d4 to $f = 1\!\!1_E$ we get the following: if Jordan measurable sets $E, F \subset \mathbb{R}^2$ satisfy $v_1(E_y) = v_1(F_y)$ for all y then $v_2(E) = v_2(F)$; here $E_y = \{x : (x, y + \sin x) \in E\}$ (and the same for F_y). But do not think that $v_1(E_y)$ is the length of the sinusoid inside E; it is not.



Here is another case: $E_r = \{\theta \in [0, 2\pi) : (r\cos\theta, r\sin\theta) \in E\}$; now $v_1(E_r)$ is the length of the circle inside E, multiplied by r; and in fact, the equality $v_1(E_r) = v_1(F_r)$ for all r implies $v_2(E) = v_2(F)$, as we'll see in Sect. 9.



Note that the parallel circles are equidistant; the parallel sinusoids are not.



We'll return to this matter in Analysis 4.

7d14 Exercise. Given $\alpha \in [0, 2\pi)$, consider the rotation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$. For an arbitrary box $B \subset \mathbb{R}^2$ prove that the rotated box $T(B)$ is Jordan measurable and $v(T(B)) = v(B)$. (You know, we did not prove rotation invariance yet...)

7d15 Exercise. [1] Consider the set $E = \{(x,y,z) : 0 \le z \le 1-x^2-y^2\} \subset \mathbb{R}^3$.

(a) Find the volume of E via $\int v_2(E^z)\,\mathrm{d}z$.

(b) Using (a) and the equality $\int v_2(E^z)\,\mathrm{d}z = \int v_1(E_{x,y})\,\mathrm{d}x\mathrm{d}y$, find the mean[2] of the function $(x,y) \mapsto 1-x^2-y^2$ on the disk $\{(x,y) : x^2+y^2 \le 1\} \subset \mathbb{R}^2$.

(c) Similarly to (a), (b), find the mean of the function $x \mapsto |x|^p$ on the ball $\{x : |x| \le 1\} \subset \mathbb{R}^n$ for $p \in (0,\infty)$.[3]

7d16 Exercise. Calculate the integral

$$\iiint_E (x_1^2 + x_2^2 + x_3^2)\, dx_1 dx_2 dx_3 \,,$$

where $E = \{(x_1,x_2,x_3) \in [0,\infty)^3 : x_1+x_2+x_3 \le a\} \subset \mathbb{R}^3$.

Answer: $a^5/20$.

7d17 Exercise. Find the volume of the intersection of two solid cylinders in $\mathbb{R}^3$: $\{x_1^2+x_2^2 \le 1\}$ and $\{x_1^2+x_3^2 \le 1\}$.

Answer: 16/3.

7d18 Exercise. Find the volume of the solid in $\mathbb{R}^3$ under the paraboloid $\{x_1^2+x_2^2 = x_3\}$ and above the square $[0,1]^2 \times \{0\}$.

Answer: 2/3.

7d19 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n\, f(x_n) = \int_0^x f(t) \frac{(x-t)^{n-1}}{(n-1)!}\, dt \,.$$

7d20 Example. Let us calculate the integral

$$\int_{[0,1]^n} \max(x_1,\dots,x_n)\, dx_1 \dots dx_n \,.$$

First of all, by symmetry, we assume that $1 \ge x_1 \ge x_2 \ge \dots \ge x_n \ge 0$, and multiply the answer by $n!$. Then $\max(x_1,\dots,x_n) = x_1$, and we get

$$n! \int_0^1 x_1\, dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n = n! \int_0^1 \frac{x_1^n\, dx_1}{(n-1)!} = \frac{n}{n+1} \,.$$

[1] Exam of 26.01.14, Question 4.

[2] Recall (6g18).

[3] Hint: you do not need the volume of the ball (nor the area of the disk)! And of course, $|x|^p$ stands for $(x_1^2 + \dots + x_n^2)^{p/2}$.

7d21 Exercise. Compute the integral $\int_{[0,1]^n} \min(x_1,\dots,x_n)\,dx_1\dots dx_n$.
Answer: $\frac{1}{n+1}$.

7d22 Exercise. Find the volume of the n-dimensional simplex

$$\{x : x_1,...,x_n \ge 0,\ x_1+...+x_n \le 1\}\,.$$

Answer: $\frac{1}{n!}$.

7d23 Exercise. Suppose the function f depends only on the first coordinate. Then

$$\int_{\mathbb{B}} f(x_1)\,dx = v_{n-1}\int_{-1}^{1} f(x_1)(1-x_1^2)^{(n-1)/2}\,dx_1\,,$$

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^n$, and v_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

The next exercises examine further a very interesting phenomenon of "concentration of high-dimensional volume" touched before, in 6h4(b); it was seen there that in high dimension the volume of a ball concentrates near the sphere,[1] and now we'll see that it also concentrates near a hyperplane![2]

7d24 Exercise. Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$, and $P=\{x\in\mathbb{B} : |x_1|<0.01\}$. What is larger, $v_n(P)$ or $v_n(\mathbb{B}\setminus P)$, if n is sufficiently large?

7d25 Exercise. Given $\varepsilon>0$, show that the quotient

$$\frac{v_n(\{x\in\mathbb{B} : |x_1|>\varepsilon\})}{v_n(\mathbb{B})}$$

tends to zero as $n\to\infty$.[3]

Could you find the asymptotic behavior of the quotient above as $n\to\infty$?

Given an integrable $f:\mathbb{R}^n\to\mathbb{R}$ and a box $B\subset\mathbb{R}^n$, we introduce $f_B:\mathbb{R}^n\to\mathbb{R}$ by

$$f_B(x)=\frac{1}{v(B)}\int_{B+x} f\,;$$

that is, $f_B(x)$ is the mean value of f on the shifted box $B+x=\{b+x : b\in B\}$. Clearly, the mapping $f\mapsto f_B$ is linear, and

$$\sup_{\mathbb{R}^n}|f_B|\le\frac{1}{v(B)}\int_B |f|\,.$$

[1] See also 7d15(c).

[2] Do you see a contradiction in these claims?

[3] Hint: the quotient equals $\frac{\int_\varepsilon^1 (1-t^2)^{(n-1)/2}\,dt}{\int_0^1 (1-t^2)^{(n-1)/2}\,dt}$.

7d26 Exercise. Prove that f_B is a continuous function.[1]

7d27 Exercise. (a) Let $n = 2$ and $B = [s_1, t_1] \times [s_2, t_2]$. For arbitrary $f \in C^0(\mathbb{R}^n)$ with bounded support, prove that $f_B \in C^1(\mathbb{R}^n)$ and

$$\frac{\partial}{\partial x_1} f_B(x_1, x_2) = \frac{1}{t_2 - s_2} \int_{[s_2,t_2]} \frac{1}{t_1 - s_1} \big(f_{x_1+t_1} - f_{x_1+s_1} \big) \, ;$$

(b) generalize (a) to arbitrary n.

7d28 Exercise. Prove that every $f \in C^0(\mathbb{R}^n)$ with bounded support is the limit of some uniformly convergent sequence of functions of $C^1(\mathbb{R}^n)$.[2]

7e Differentiation under the integral sign

Integration of the function $x \mapsto \int f(x, \cdot)$ is useful, but differentiation of this function is also widely used. Imagine for instance that a function depends on time: $f(x, t)$. Then its integral depends on time, too: $t \mapsto \int f(x, t) \, \mathrm{d}x$. According to the so-called Leibniz integral rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f(x, t) \, \mathrm{d}x = \int \Big(\frac{\partial}{\partial t} f(x, t) \Big) \mathrm{d}x$$

under appropriate conditions.

Instead of differentiating $\int f(x, t) \, \mathrm{d}x$ we'll integrate $\int \big(\frac{\partial}{\partial t} f(x, t) \big) \mathrm{d}x$; this little trick shifts the work onto the iterated integral theorem!

7e1 Theorem. Let $B \subset \mathbb{R}^n$ be a box, and $f, g : B \times [0, 1] \to \mathbb{R}$ Lipschitz functions such that $f'_x(t) = g_x(t)$ for all $x \in B$, $t \in (0, 1)$. Then $F'(t) = G(t)$ for all $t \in (0, 1)$, where $F(t) = \int_B f(x, t) \, \mathrm{d}x$ and $G(t) = \int_B g(x, t) \, \mathrm{d}x$.

Clarification. By "$F'(t) = G(t)$" we mean that the derivative exists and equals $G(t)$; and "$f'_x(t) = g_x(t)$" is interpreted similarly.

Proof. We know (recall Sect. 7b) that F and G are Lipschitz continuous. It is sufficient to prove that $\int_0^t G(s) \, \mathrm{d}s = F(t) - F(0)$ for all $t \in (0, 1)$. We have $f_x(t) - f_x(0) = \int_0^t g_x(s) \, \mathrm{d}s$, therefore

$$F(t) - F(0) = \int_B \big(f(x, t) - f(x, 0) \big) \, \mathrm{d}x = \int_B \mathrm{d}x \int_0^t \mathrm{d}s \, g(x, s) =$$
$$= \int_0^t \mathrm{d}s \int_B \mathrm{d}x \, g(x, s) = \int_0^t \mathrm{d}s \, G(s) \, .$$

□

[1] Hint: All $[f]$ such that f_B is continuous are a closed set in the normed space of equivalence classes.

[2] Hint: consider f_B for a small B close to 0.

7e2 Exercise. [1] Consider the function

$$F(t) = \int_0^{\pi/2} \ln\bigl(t^2 - \sin^2 x\bigr)\, \mathrm{d}x \quad \text{for } t > 1\,.$$

(a) Write $F'(t)$ as an integral in x; substituting $\tan x = u$ prove that

$$F'(t) = \frac{\pi}{\sqrt{t^2-1}} = \pi \frac{\mathrm{d}}{\mathrm{d}t} \ln\bigl(t + \sqrt{t^2-1}\bigr)\,;$$

(b) for $t \to +\infty$ prove that $F(t) = \pi \ln t + o(1)$ and $\pi \ln\bigl(t + \sqrt{t^2-1}\bigr) = \pi \ln(2t) + o(1)$;

(c) prove that

$$F(t) = \pi \ln \frac{t + \sqrt{t^2-1}}{2}\,.$$

The conditions of Th. 7e1 can be relaxed in several aspects. First, it is easy to replace "Lipschitz" with "uniformly continuous" (since a uniformly continuous function is the uniform limit of Lipschitz functions).

7e3 Exercise. (a) If $f \in C^1(\mathbb{R}^n)$ has a bounded support, then $f_B \in C^2(\mathbb{R}^n)$ and

$$D_i f_B = (D_i f)_B \quad \text{for } i = 1, \dots, n\,;$$

(b) every $f \in C^0(\mathbb{R}^n)$ with bounded support is the limit of some uniformly convergent sequence of functions of $C^2(\mathbb{R}^n)$;

(c) the same as (b), but replace $C^2(\mathbb{R}^n)$ with $C^k(\mathbb{R}^n)$, $k = 1, 2, 3, \dots$

Prove it.[2]

By more effort it is possible (but not easy) to substantially relax the conditions of Th. 7e1, as sketched below.

7e4 Remark. Let $E \subset \mathbb{R}^n$ be a Jordan set; $f, g : E \times (0,1) \to \mathbb{R}$ integrable functions; and $\frac{\partial}{\partial t} f(x,t) = g(x,t)$ for all $(x,t) \in E \times (0,1)$.[3] Then

(a) the function $f(\cdot, t)$ is integrable on E for all $t \in (0,1)$;

(b) the function $F : t \mapsto \int_E f(\cdot, t)$ is differentiable on $(0,1)$, and ${}_*\!\int_E g(\cdot, t) \le F'(t) \le {}^*\!\int_E g(\cdot, t)$ for all $t \in (0,1)$.

---

[1] Zorich, Sect. 17.1.3, Example 4.

[2] Hint: use 7d27(b) and 7e1.

[3] That is, $f(x, \cdot)$ must be differentiable; but maybe not continuously differentiable.

For now, we could prove (a) [1] and the equality $F(s) - F(r) = \int_r^s G(t)\,\mathrm{d}t$ for $0 < r < s < 1$, where ${}_*\!\int_E g(\cdot, t) \le G(t) \le {}^*\!\int_E g(\cdot, t)$, but not (b).[2]

Higher generality, achieved within Lebesgue's integration theory, is beyond our course even in formulation.[3]

Still more can be said using the so-called gauge integral.[4,5]

[1] By 6b28(a) and the mean value theorem, ${}_*\!\int_r^s g(x,t)\,\mathrm{d}t \le f(x,s) - f(x,r) \le {}^*\!\int_r^s g(x,t)\,\mathrm{d}t$ for $x \in E$ and $0 < r < s < 1$. By Th. 7d1 the functions $x \mapsto {}_*\!\int_r^s g(x,t)\,\mathrm{d}t$ and $x \mapsto {}^*\!\int_r^s g(x,t)\,\mathrm{d}t$ are equivalent integrable functions. Thus we get integrability of $f(\cdot, s) - f(\cdot, r)$ whenever $0 < r < s < 1$. In order to prove integrability of $f(\cdot, t)$ for all t it remains to prove it for a single t. This is easy: by Th. 7d1 (again), the functions $t \mapsto {}_*\!\int_E f(\cdot, t)$ and $t \mapsto {}^*\!\int_E f(\cdot, t)$ are equivalent integrable functions; they must be equal at some t (at least one).

[2] The proof of (b) uses the following fact. Let a sequence of integrable functions $f_i : E \to [0, 1]$ converge pointwise to a function $f : E \to [0, 1]$. Then the sequence of numbers $\int_E f_i$ converges to a number that belongs to $[\,{}_*\!\int_E f, {}^*\!\int_E f]$.

[3] Let $E \subset \mathbb{R}^n$ be a measurable set; $f : E \times (0, 1) \to \mathbb{R}$ a integrable function; $\frac{\partial}{\partial t} f(x, t) = g(x, t)$ exists for all $(x, t) \in E \times (0, 1)$; and $\int_E \sup_t |g(\cdot, t)| < \infty$. Then (a) $f(\cdot, t)$ and $g(\cdot, t)$ are measurable for all $t \in (0, 1)$, and (b) $\frac{\mathrm{d}}{\mathrm{d}t} \int_E f(\cdot, t) = \int_E g(\cdot, t)$ for all $t \in (0, 1)$. (All integrals are Lebesgue integrals; also measurability and integrability are Lebesgue's.)

[4] "Necessary and sufficient conditions for differentiating under the integral sign" by Erik Talvila. Amer. Math. Monthly, June/July 2001, **108**, 432–436.

[5] "[...] analysis is full of statements that are easy theorems under restrictive hypotheses, and harder theorems under more general hypotheses [...] Statements about interchanging limiting operations (e.g. differentiation under the integral) are classic examples; the truth boundary is so often unknown — or it is so difficult or unrewarding to formulate useful "if and only if" conditions under which the statement is a theorem, that nobody bothers to do it." MathStackExchange:Why are gauge integrals not more popular?.