8 A glimpse into Lebesgue's theory

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Lebesgue's criterion for Riemann integrability is proved. Riemann integral is extended to semicontinuous functions, and Jordan measure to open sets and closed sets (not always Jordan measurable).

8a What is the problem

Consider a bounded function $f : (0,1) \to \mathbb{R}$. If f is continuous then it is integrable (even if it is not uniformly continuous, like $\sin(1/x)$). A step function is (generally) discontinuous, and still, integrable; its set of discontinuity points is finite. Non-integrable functions mentioned in 6b29 are "very discontinuous", having intervals of discontinuity points. The function of 6b31 (or 6c5) has a dense set of discontinuity points, and still, is integrable. Can integrability be decided via the set of discontinuity points? An affirmative answer was given by Lebesgue, it involves the notion of Lebesgue measure zero (rather than volume zero).

"This aesthetically pleasing integrability criterion has little practical value" (Bichteler).[1] Well, if you use it when proving simple facts, such as integrability of $\sqrt[3]{f}$ or fg (for integrable f and g), you may find far more elementary proofs. But here is a harder case. The so-called improper integral will be applied to unbounded functions f such that the function

$$\operatorname{mid}(-M, f, M) : x \mapsto \begin{cases} -M & \text{when } f(x) \le -M, \\ f(x) & \text{when } -M \le f(x) \le M, \\ M & \text{when } M \le f(x) \end{cases}$$

is integrable for all $M > 0$. The sum of two such functions is also such function. This fact follows easily from Lebesgue's criterion. You may discover another proof, but I doubt it will be simpler!

[1] From book "Integration — a functional approach" by Klaus Bichteler (1998); see Exercise 6.16 on p. 27.

Two techniques of this section are also of independent interest and will be reused: derivative of a set function; extended integral and measure.

Lebesgue theory extends the Riemann integral to a very wide class of functions, and Jordan measure to a very wide class of sets. This is far beyond our needs. However, we want to measure (at least) open sets and closed sets (even if not Jordan measurable). The relevant portion of Lebesgue's extension is easy to describe via lower and upper Riemann integral (for functions) and inner and outer Jordan measure (for sets).

A natural quantitative measure of nonintegrability is the difference

$$A = \int^*_{(0,1)} f - \int_{*(0,1)} f \ \in [0,\infty) \, .$$

What about a natural quantitative measure of discontinuity of f? At a given point $x_0 \in (0,1)$ it is the difference[1]

$$\operatorname{Osc}_f(x_0) = \limsup_{x \to x_0} f(x) - \liminf_{x \to x_0} f(x) \ \in [0,\infty) \, .$$

But it depends on x_0. In order to get a number we integrate the oscillation function:

$$B = \int^*_{(0,1)} \operatorname{Osc}_f \, .$$

We would be happy to know that $B = 0 \implies A = 0$, even happier to know that $B = 0 \iff A = 0$, but here is a surprise:

$$A = B \, .$$

Qualitatively,

$$(f \text{ is integrable}) \iff (\operatorname{Osc}_f \text{ is negligible}) \, .$$

And of course, we need a multidimensional theory; $(0,1)$ is only the simplest case.

It may seem that the equality $A = B$ is an easy matter, just

$$B = \int \operatorname{Osc}_f = \int \big((\limsup f) - (\liminf f) \big) = \\ = \int \limsup f - \int \liminf f = \int^* f - \int_* f \, .$$

However, Osc_f is generally nonintegrable; and upper integral is not linear. Quite a problem! Does it invalidate the equality $A = B$? Fortunately, it does not. Rather, a finer argument is needed.

[1] The case "$x = x_0$" is included into "$x \to x_0$" in this introductory section for simplicity, but in next sections it is excluded, according to the standard notation.

8b Semicontinuous envelopes

Given a locally bounded $f : \mathbb{R}^n \to \mathbb{R}$, we introduce its *upper envelope* (called also upper semicontinuous envelope, or upper semicontinuous majorant) $f^* : \mathbb{R}^n \to \mathbb{R}$ by

$$f^*(x_0) = \max\Big(f(x_0), \limsup_{x\to x_0} f(x)\Big) = \inf_{\delta>0} \sup_{|x-x_0|<\delta} f(x)\,. \tag{8b1}$$

8b2 Exercise. (a) $\sup_{B^\circ} f^* = \sup_{B^\circ} f$ for every box $B \subset \mathbb{R}^n$; prove it.

(b) It may happen that $\sup_{\overline{B}} f^* > \sup_{\overline{B}} f$; find an example.

By 8b2(a) and 6c4,

$$\overline{\int_B} f^* = \overline{\int_B} f \quad \text{for every box } B\,. \tag{8b3}$$

Similarly, the *lower envelope* f_*, defined by

$$f_*(x_0) = \min\Big(f(x_0), \liminf_{x\to x_0} f(x)\Big) = \sup_{\delta>0} \inf_{|x-x_0|<\delta} f(x)\,, \tag{8b4}$$

satisfies

$$\underline{\int_B} f_* = \underline{\int_B} f \quad \text{for every box } B\,. \tag{8b5}$$

Also, $(-f)^* = -f_*$.

8b6 Exercise. Let f be integrable on B.

(a) Functions f_*, f, f^* are equivalent on B; prove it.

(b) However, these functions need not be equal on B°; find an example.

8b7 Exercise. For every $A \subset \mathbb{R}^n$,

(a) $(1\!\!1_A)^* = 1\!\!1_{\overline{A}}$ and $(1\!\!1_A)_* = 1\!\!1_{A^\circ}$;

(b) if A is bounded, then $v^*(A) = v^*(\overline{A})$ and $v_*(A) = v_*(A^\circ)$.

Prove it.

An envelope is its own envelope:

$$(f_*)_* = f_*\,; \quad (f^*)^* = f^*\,. \tag{8b8}$$

(The latter holds since, similarly to 8b2(a), $\sup_{|x-x_0|<\delta} f^*(x) = \sup_{|x-x_0|<\delta} f(x)$.)

A function h such that $h^* = h$ is called *upper semicontinuous;* and g such that $g_* = g$ is called *lower semicontinuous.* Clearly, h is upper semicontinuous if and only if $(-h)$ is lower semicontinuous.

8b9 Example. A semicontinuous function need not be integrable. Here is a counterexample.

We take a sequence of pairwise disjoint closed intervals $[s_1,t_1],[s_2,t_2],\dots \subset (0,1)$ such that $\sum_k (t_k - s_k) = a < 1$ and the open set $G = (s_1,t_1)\cup(s_2,t_2)\cup\dots$ is dense in $(0,1)$.[1] A function $f = 1\!\!1_G$ is lower semicontinuous (by 8b7(a): $f_* = 1\!\!1_{G^\circ} = 1\!\!1_G = f$), and $f^* = 1\!\!1_{[0,1]}$ (by 8b7(a) again: $f^* = 1\!\!1_{\overline{G}} = 1\!\!1_{[0,1]}$). We have ${}_*\!\int f = a$ (it cannot be larger, since every box B such that $\inf_B f = 1$ is contained in some (s_k,t_k)),[2] but ${}^*\!\int f = 1$ (since $f^* = 1\!\!1_{[0,1]}$).

8b10 Example. Upper integral is not additive on lower semicontinuous functions. That is, ${}^*\!\int (f_1+f_2) < {}^*\!\int f_1 + {}^*\!\int f_2$ for some lower semicontinuous f_1, f_2.

We use $G = (s_1,t_1)\cup(s_2,t_2)\cup\dots$ from 8b9, introduce open sets $G_1 = (s_1,t_1)\cup(s_3,t_3)\cup\dots$ and $G_2 = (s_2,t_2)\cup(s_4,t_4)\cup\dots$, then $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \emptyset$. We have $\overline{G}_1 \cup \overline{G}_2 = \overline{G_1 \cup G_2} = \overline{G} = [0,1]$, thus[3] $\overline{G}_1 = [0,1]\setminus G_2$ and $\overline{G}_2 = [0,1]\setminus G_1$. Lower semicontinuous functions $f_1 = 1\!\!1_{G_1}$, $f_2 = 1\!\!1_{G_2}$ satisfy $f_1 + f_2 = f$, ${}_*\!\int f_1 = a_1 = \sum_k (t_{2k-1} - s_{2k-1})$, ${}_*\!\int f_2 = a_2 = \sum_k (t_{2k} - s_{2k})$, $a_1 + a_2 = a$. On the other hand, $f_1^* = 1\!\!1_{\overline{G}_1} = 1\!\!1_{[0,1]\setminus G_2} = 1\!\!1_{[0,1]} - 1\!\!1_{G_2} = 1\!\!1_{[0,1]} - f_2$, therefore ${}^*\!\int f_1 = {}^*\!\int f_1^* = {}^*\!\int (1\!\!1_{[0,1]} - f_2) = 1 - {}_*\!\int f_2 = 1 - a_2$ (since $U(1\!\!1_{[0,1]} - f_2, P) = 1 - L(f_2, P)$ for all partitions P of $[0,1]$). Similarly, ${}^*\!\int f_2 = 1 - a_1$. Finally, ${}^*\!\int f_1 + {}^*\!\int f_2 = (1-a_2) + (1-a_1) = 2 - a$, but ${}^*\!\int (f_1+f_2) = {}^*\!\int f = 1 < 2 - a$.

Taking $(-f_1), (-f_2)$ we see that lower integral is not additive on upper semicontinuous functions.

Interestingly, ${}_*\!\int (f_1+f_2) = {}_*\!\int f = a = a_1 + a_2 = {}_*\!\int f_1 + {}_*\!\int f_2$ in 8b10. Maybe, lower integral is additive on lower semicontinuous functions (and, equivalently, upper integral is additive on upper semicontinuous functions)? Yes, it is!

8b11 Proposition. ${}^*\!\int (f+g) = {}^*\!\int f + {}^*\!\int g$ for all upper semicontinuous bounded functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

The proof will be given in the end of Sect. 8c.

8c Differentiating set functions

As was noted in the end of Sect. 6a, in dimension one an (ordinary) function $\tilde{F} : \mathbb{R} \to \mathbb{R}$ leads to a box function $F : [s,t) \mapsto \tilde{F}(t) - \tilde{F}(s)$; clearly, F

[1] Its complement $[0,1]\setminus G$ is sometimes called a fat Cantor set.

[2] See also 8e9, 8e10.

[3] Between two intervals of G_1 there is an interval of G_2, and vice versa.

is additive: $F([r,s)) + F([s,t)) = F([r,t))$. Moreover, every additive box function F defined on one-dimensional boxes corresponds to some $\tilde F$ (unique up to adding a constant); namely, $\tilde F(t) = F([0,t))$ (for $t > 0$).

If $\tilde F$ is differentiable, $\tilde F' = f$, then F and f are related by

$$\frac{F([t-\varepsilon,t))}{\varepsilon} \to f(t)\,, \quad \frac{F([t,t+\varepsilon))}{\varepsilon} \to f(t) \quad \text{as } \varepsilon \to 0+ \ .$$

Equivalently,

$$\text{(8c1)} \qquad \frac{F([t-\varepsilon_1,t+\varepsilon_2))}{\varepsilon_1+\varepsilon_2} \to f(t) \qquad \text{as } \varepsilon_1,\varepsilon_2 \to 0+ \ .$$

And if f is integrable on $[s,t]$ then[1]

$$F([s,t)) = \int_{[s,t]} f \,.$$

In dimension 2 a similar construction exists, but is more cumbersome and less useful:

$$F([s_1,t_1)\times[s_2,t_2)) = \tilde F(t_1,t_2) - \tilde F(t_1,s_2) - \tilde F(s_1,t_2) + \tilde F(s_1,s_2)\,;$$
$$\tilde F(s,t) = F([0,s)\times[0,t)) \quad \text{for } s,t>0 \ ;$$

this time $\tilde F$ is unique up to adding $\varphi(t_1)+\psi(t_2)$. In higher dimensions $\tilde F$ is even less useful; we do not need it. Instead, we generalize (8c1) as follows.

8c2 Definition. A number a is the *derivative* at $x_0 \in \mathbb{R}^n$ of a box function F, if

$$\forall \varepsilon>0\ \exists \delta>0\ \forall B \ni x_0 \ \biggl(\max_{x\in B} |x-x_0| < \delta \quad \Longrightarrow \quad \Bigl| \frac{F(B)}{v(B)} - a \Bigr| < \varepsilon \biggr)\,.$$

Clearly, such a is unique. If it exists, we say that F is differentiable at x_0 and write $F'(x_0) = a$. Less formally,

$$F'(x) = \lim_{B\to x} \frac{F(B)}{v(B)}\,.$$

In general the limit need not exist, and we introduce the lower and upper derivatives,

$${}_*F'(x) = \liminf_{B\to x} \frac{F(B)}{v(B)}\,, \quad {}^*F'(x) = \limsup_{B\to x} \frac{F(B)}{v(B)}\,.$$

[1] Can you prove it (a) for continuous f, (b) in general? Try 6b28(a) in concert with the mean value theorem. Anyway, it is the one-dimensional case of (8c10).

More formally,

$$
\text{(8c3)}\qquad
\begin{aligned}
{}_*F'(x_0) &= \sup_{\delta>0} \inf_{B\ni x_0 : \sup_{x\in B}|x-x_0|<\delta} \frac{F(B)}{v(B)}\,, \\
{}^*F'(x_0) &= \inf_{\delta>0} \sup_{B\ni x_0 : \sup_{x\in B}|x-x_0|<\delta} \frac{F(B)}{v(B)}\,.
\end{aligned}
$$

8c4 Exercise. For two box functions F and G,

(a) if F and G are differentiable, then $(F+G)' = F' + G'$;

(b) if G is differentiable, then ${}_*(F+G)' = {}_*F' + G'$ and ${}^*(F+G)' = {}^*F' + G'$;

(c) generally, ${}_*(F+G)' \ge {}_*F' + {}_*G'$ and ${}^*(F+G)' \le {}^*F' + {}^*G'$.

Prove it.

8c5 Remark. The restriction $B \ni x_0$ is not needed when proving 8b11. It is stipulated only in order to conform to (8c1). Without this restriction, in the one-dimensional case, ${}_*F', {}^*F'$ are the envelopes of $\tilde F'$ (assuming differentiability of $\tilde F$), and so, F is differentiable if and only if $\tilde F$ is *continuously* differentiable. The restriction $B \ni x_0$ ensures that F is differentiable if and only if $\tilde F$ is differentiable (including such cases as (2d2)).

An integrable function f leads to a box function $F : B \mapsto \int_B f$. Can we restore f from F by differentiation, $f = F'$? Generally, we cannot, since equivalent functions f_1, f_2 lead to the same F.

8c6 Exercise. (a) If a locally bounded $f : \mathbb{R}^n \to \mathbb{R}$ is upper semicontinuous and $F : B \mapsto {}^*\!\!\int_B f$ then ${}^*F' \le f$; prove it. It can happen that ${}^*F' \ne f$; find an example.

(b) If f is continuous then $F : B \mapsto \int_B f$ is differentiable, and $F' = f$; prove it.

Recall that a box function F is called *additive* if $F(B) = \sum_{C\in P} F(C)$ for every box B and every partition P of B. Also, F is *subadditive* if $F(B) \le \sum_{C\in P} F(C)$, and *superadditive* if $F(B) \ge \sum_{C\in P} F(C)$.

8c7 Lemma. If a superadditive box function F satisfies ${}_*F'(x) \ge 0$ for all $x \in \overline{B}_0$ (B_0 being a given box), then $F(\overline{B}_0) \ge 0$.

Proof. Assume the contrary: $F(\overline{B}_0) < 0$. We take a partition P_0 of B_0 such that $\forall C \in P_0\ \operatorname{diam}(C) \le 1$. Taking into account that

$$
\frac{F(\overline{B}_0)}{v(B_0)} \ge \frac{1}{v(B_0)} \sum_{C\in P_0} F(\overline{C}) = \sum_{C\in P_0} \frac{F(\overline{C})}{v(C)} \frac{v(C)}{v(B_0)} \ge \Big(\min_{C\in P_0} \frac{F(\overline{C})}{v(C)} \Big) \underbrace{\sum_{C\in P_0} \frac{v(C)}{v(B_0)}}_{=1}
$$

we take $B_1 \in P_0$ such that

$$\frac{F(\overline{B}_1)}{v(B_1)} \le \frac{F(\overline{B}_0)}{v(B_0)} \, .$$

Similarly, we take a partition P_1 of B_1 such that $\forall C \in P_1 \;\; \operatorname{diam}(C) \le 1/2$ and find $B_2 \in P_1$ such that

$$\frac{F(\overline{B}_2)}{v(B_2)} \le \frac{F(\overline{B}_1)}{v(B_1)} \le \frac{F(\overline{B}_0)}{v(B_0)} \, .$$

Continuing this way we get a sequence $(B_i)_i$ of boxes such that

$$\overline{B}_0 \supset \overline{B}_1 \supset \overline{B}_2 \dots \, , \quad \operatorname{diam}(B_i) \to 0 \, , \quad \text{and} \quad \forall i \;\; \frac{F(\overline{B}_i)}{v(B_i)} \le \frac{F(\overline{B}_0)}{v(B_0)} \, .$$

We take x_0 such that

$$\forall i \;\; x_0 \in \overline{B}_i \, ,$$

and get a contradiction:

$${}_*F'(x_0) \le \frac{F(\overline{B}_0)}{v(B_0)} < 0$$

since $\forall \delta > 0 \; \exists k \;\; \inf_{B \ni x_0 : \max_{x \in B} |x - x_0| < \delta} \frac{F(B)}{v(B)} \le \frac{F(\overline{B}_i)}{v(B_i)}$. □

By the way, the one-dimensional case of Lemma 8c7 can be used for proving (2d5), as follows.

8c8 Exercise. Given a path $\gamma : [t_0, t_1] \to X$, differentiable on (t_0, t_1), in an n-dimensional normed space X, consider a box function

$$F([s,t]) = M(t-s) - \|\gamma(t) - \gamma(s)\| \quad \text{for } t_0 < s < t < t_1 \, ,$$

where $M = \sup_{t \in (t_0, t_1)} \|\gamma'(t)\|$. Prove that F is superadditive and ${}_*F'(t) = M - \|\gamma'(t)\| \ge 0$ for all $t \in (t_0, t_1)$. Applying 8c7 get (2d5).[1]

8c9 Exercise. (a) If an additive box function F is differentiable on a box B then

$$v(B) \inf_{x \in B} F'(x) \le F(B) \le v(B) \sup_{x \in B} F'(x) \, .$$

(b) For every additive box function F,

$$v(B) \inf_{x \in B} {}_*F'(x) \le F(B) \le v(B) \sup_{x \in B} {}^*F'(x) \, .$$

Prove it.[2]

[1] Mind the endpoints; F need not be differentiable at t_0 and t_1.

[2] Hint: similarly to 8c8, try the box function $Mv - F$.

Combining 8c9(a) and 6b28(a) we get

$$F(B) = \int_B F' \tag{8c10}$$

whenever F' exists and is integrable on B. Here is a more general result.

8c11 Exercise. Prove that

$$\mathop{{}_*\!\!\int}_B {}_*F' \le F(B) \le \mathop{{}^*\!\!\int}_B {}^*F'$$

for every box B and additive box function F such that ${}_*F'$ and ${}^*F'$ are bounded on B.[1]

***Proof** of Prop. 8b11.* By (6d9) it is sufficient to prove that ${}^*\!\int(f+g) \ge {}^*\!\int f + {}^*\!\int g$. We consider two additive box functions, $F : B \mapsto {}^*\!\int_B f$ and $G : B \mapsto {}^*\!\int_B g$. By 8c6(a), ${}^*F' \le f$ and ${}^*G' \le g$. By 8c4(c), ${}^*(F+G)' \le {}^*F' + {}^*G'$. Thus, ${}^*(F+G)' \le f + g$.

On the other hand, for every continuous function $h : \mathbb{R}^n \to \mathbb{R}$ and the corresponding box function $H : B \mapsto \int_B h$ we have $H' = h$ by 8c6(b). If $h \ge f+g$, then $H' \ge {}^*(F+G)'$. By 8c4(b), ${}^*(F+G-H)' = {}^*(F+G)' - H' \le 0$, that is, ${}_*(H-F-G)' \ge 0$. By 8c7, $(H-F-G)(\overline{B}) \ge 0$, that is,

$$\int_B h \ge \mathop{{}^*\!\!\int}_B f + \mathop{{}^*\!\!\int}_B g$$

for every continuous $h \ge f+g$. Taking the infimum over all such h we get in the right-hand side ${}^*\!\int_B (f+g)$ by (6f6). It remains to take B large enough. □

8d Oscillation and integrability

8d1 Definition. The *oscillation function* $\mathrm{Osc}_f : \mathbb{R}^n \to \mathbb{R}$ of a locally bounded $f : \mathbb{R}^n \to \mathbb{R}$ is the difference of envelopes,

$$\mathrm{Osc}_f = f^* - f_* \,.$$

8d2 Proposition. ${}^*\!\int f - {}_*\!\int f = {}^*\!\int \mathrm{Osc}_f$ for all bounded $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

[1] Hint: look again at 6b28(a), and use 8c9(b).

Proof. By 8b11, (6b24), (8b3) and (8b5),

$$\begin{aligned}\int^* \operatorname{Osc}_f = \int^* \big(f^* + (-f_*)\big) = \int^* f^* + \int^* (-f_*) = \\ = \int^* f^* - \int_* f_* = \int^* f - \int_* f \,.\end{aligned}$$

$\square$

8d3 Corollary. A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if Osc_f is negligible.

8d4 Corollary. For every bounded $A \subset \mathbb{R}^n$,
(a) $v^*(A) - v_*(A) = v^*(\partial A)$;
(b) A is Jordan measurable if and only if ∂A is of volume zero.

Proof. $\operatorname{Osc}_{\mathbb{1}_A} = (\mathbb{1}_A)^* - (\mathbb{1}_A)_* = \mathbb{1}_{\overline{A}} - \mathbb{1}_{A^\circ} = \mathbb{1}_{\partial A}$; by 8d2, $\int^* \mathbb{1}_A - \int_* \mathbb{1}_A = \int^* \mathbb{1}_{\partial A}$, which means (a); (b) follows. $\square$

Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}$ a bounded function. Then Osc_f (as well as f_*, f^*) is well-defined on E°.

If in addition E is Jordan measurable, then integrability of f on E is well-defined (recall the end of Sect. 6g), it is integrability on $\mathbb{R}^n$ of the function[1]

$$f \cdot \mathbb{1}_E : x \mapsto \begin{cases} f(x) & \text{for } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

By 8d3, this integrability is equivalent to negligibility of $\operatorname{Osc}_{f\cdot\mathbb{1}_E}$. Note that

$$\operatorname{Osc}_{f\cdot\mathbb{1}_E} = \begin{cases} \operatorname{Osc}_f & \text{on } E^\circ, \\ \text{something bounded} & \text{on } \partial E, \\ 0 & \text{outside } \overline{E}. \end{cases}$$

Taking into account that ∂E is of volume zero by 8d4(b) we see that $\operatorname{Osc}_{f\cdot\mathbb{1}_E}$ is equivalent to $\operatorname{Osc}_f \cdot \mathbb{1}_{E^\circ}$. Thus,

$$\text{(8d5)} \qquad (f \text{ is integrable on } E) \iff (\operatorname{Osc}_f \text{ is negligible on } E^\circ)\,.$$

If the set $\{x : \operatorname{Osc}_f(x) \ne 0\}$ is of volume zero, then Osc_f is negligible by 6g13, thus f is integrable. However, an integrable function can be discontinuous on a dense set; for example, see 6b31 (or 6c5).

[1]This use of the notation $f \cdot \mathbb{1}_E$ is a convention (abuse of language), of course.

8d6 Remark. It is tempting to invent an appropriate notion "negligible set" such that[1]

(a) f is negligible if and only if $\{x : f(x) \ne 0\}$ is negligible,

and therefore

(b) f is integrable if and only if $\{x : \operatorname{Osc}_f(x) \ne 0\}$ is negligible.

Is this possible? Yes and no...

Bad news: it can happen that $\{x : f(x) \ne 0\} = \{x : g(x) \ne 0\}$, f is negligible, but g is not.

Good news: it cannot happen that $\{x : \operatorname{Osc}_f(x) \ne 0\} = \{x : \operatorname{Osc}_g(x) \ne 0\}$, f is integrable, but g is not.

That is, (b) succeeds, but not due to (a). Rather, (b) succeeds in spite of the fact that (a) fails. A paradox? Here is an explanation: Osc_f is not just a function; it is an upper semicontinuous function. For upper semicontinuous f, g it cannot happen that $\{x : f(x) \ne 0\} = \{x : g(x) \ne 0\}$, f is negligible, but g is not.

8e Extending Riemann integral and Jordan measure

As we know, on upper semicontinuous functions, upper integral is additive, and lower integral is not. Thus, it is reasonable to extend the integral, defining[2]

$$ {}^{\mathrm{e}}\!\!\int f = {}^{*}\!\!\int f \quad \text{for upper semicontinuous } f\,. $$

(The small "e" stands for "extended".) But what about linearity? First, by (6d6), ${}^{\mathrm{e}}\!\!\int cf = c\,{}^{\mathrm{e}}\!\!\int f$ for $c \ge 0$; in order to get it for $c < 0$ we define

$$ {}^{\mathrm{e}}\!\!\int f = {}_{*}\!\!\int f \quad \text{for lower semicontinuous } f\,. $$

Second, what about ${}^{\mathrm{e}}\!\!\int (f+g)$ when f is upper semicontinuous and g is lower semicontinuous? Equivalently, ${}^{\mathrm{e}}\!\!\int (f-g)$ when f, g are upper semicontinuous? We can define

$$ {}^{\mathrm{e}}\!\!\int (f-g) = {}^{*}\!\!\int f - {}^{*}\!\!\int g \quad \text{for upper semicontinuous } f, g $$

if this is correct; that is, we need

$$ {}^{*}\!\!\int f_1 - {}^{*}\!\!\int g_1 = {}^{*}\!\!\int f_2 - {}^{*}\!\!\int g_2 \quad \text{whenever } f_1 - g_1 = f_2 - g_2 $$

[1] Assuming that f is bounded, with bounded support, of course.

[2] As before, $f : \mathbb{R}^n \to \mathbb{R}$ is bounded, with bounded support.

for upper semicontinuous f_1, f_2, g_1, g_2. Well, this is a simple trick:

(8e1)
$$f_1-g_1 = f_2-g_2 \implies f_1+g_2 = f_2+g_1 \implies \int^* (f_1+g_2) = \int^* (f_2+g_1) \implies$$
$$\implies \int^* f_1 + \int^* g_2 = \int^* f_2 + \int^* g_1 \implies \int^* f_1 - \int^* g_1 = \int^* f_2 - \int^* g_2 \,.$$

The extended integral is defined and linear (think, why) on the vector space DBSC ("Differences of Bounded SemiContinuous") of functions.[1] Note that[2]

(8e2)
$$\begin{aligned} f \ge 0 &\implies \int^{\mathrm{e}} f \ge 0\,, \\ f \ge g &\implies \int^{\mathrm{e}} f \ge \int^{\mathrm{e}} g \end{aligned} \qquad \text{for all } f, g \in \text{DBSC}\,.$$

We extend the Jordan measure accordingly (generalizing 6g1):

$${}^{\mathrm{e}}v(A) = \int^{\mathrm{e}} 1\!\!1_A \quad \text{whenever } 1\!\!1_A \in \text{DBSC}\,.$$

By 8b7(a), $1\!\!1_A$ is upper semicontinuous if A is closed, and lower semicontinuous if A is open. Thus,

$$\begin{aligned} {}^{\mathrm{e}}v(K) &= v^*(K) \quad \text{for compact } K \subset \mathbb{R}^n\,, \\ {}^{\mathrm{e}}v(G) &= v_*(G) \quad \text{for open bounded } G \subset \mathbb{R}^n\,. \end{aligned}$$

Indeed, such sets need not be Jordan measurable. (Recall G of 8b9; $v_*(G) = a < 1$, $v^*(G) = 1$.) Moreover, a planar domain *diffeomorphic* to a disk need not be Jordan measurable.[3]

Also, for $A = K \cap G$ we have $1\!\!1_A \in$ DBSC, since $1\!\!1_A = 1\!\!1_K - 1\!\!1_{K\setminus G}$; thus,

$${}^{\mathrm{e}}v(K \cap G) = v^*(K) - v^*(K \setminus G) \quad \text{for compact } K \text{ and open } G.$$

(And what about $K \cup G$?) Note that

(8e3)
$$A \subset B \implies {}^{\mathrm{e}}v(B) - {}^{\mathrm{e}}v(A) = {}^{\mathrm{e}}v(B \setminus A) \ge 0\,,$$

(8e4)
$${}^{\mathrm{e}}v(A \cup B) \le {}^{\mathrm{e}}v(A) + {}^{\mathrm{e}}v(B)$$

[1] Infinite-dimensional vector space, of course. If you want to know more on DBSC, see article "On differences of semi-continuous functions" by F. Chaatit and H.P. Rosenthal, arXiv:math/9901134. In fact, $\forall f, g \in$ DBSC $\min(f,g), \max(f,g), fg \in$ DBSC; compare it with 6f8(b) and 6f9(b).

[2] The former follows from (6d4), the latter from the former (and linearity).

[3] The Riemann mapping theorem is instrumental. See Sect. 18.8 "Change of variables" in book: D.J.H. Garling, "A course in mathematical analysis", vol. 2 (2014).

whenever $1\!\!1_A, 1\!\!1_B \in$ DBSC.

In this course we never need to integrate functions outside DBSC. If you need to integrate more bizarre functions on $\mathbb{R}^n$, try a more advanced integration theory. Several theories are available.[1] Gauge integration[2] is quite successful in one dimension, but less successful in higher dimensions.[3] Lebesgue integration theory is the most important one.[4]

8e5 Definition. For a bounded set $A \subset \mathbb{R}^n$,

$$\begin{aligned} m_*(A) &= \sup_{K \subset A} v^*(K) \quad \text{is the inner Lebesgue measure,} \\ m^*(A) &= \inf_{G \supset A} v_*(G) \quad \text{is the outer Lebesgue measure} \end{aligned}$$

(here K runs over compact sets, and G over open bounded sets); if these are equal, then A is *Lebesgue measurable,* and its *Lebesgue measure* is

$$m(A) = m_*(A) = m^*(A) \,.$$

In fact, Lebesgue non-measurable sets are *extremely* exotic. Their existence can be proved using the choice axiom, but a specific[5] example cannot be found!

We do not dive into the sea of exotic sets; here is what we need.

8e6 Lemma. Every open bounded set is Lebesgue measurable. That is,[6]

$$v_*(G) = \sup_{K \subset G} v^*(K) \quad \text{for every open bounded } G \subset \mathbb{R}^n \,,$$

the supremum being taken over all compact subsets of G.

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[1]See book "Theory of the integral" by Brian Thomson. (An excerpt: "Definition 1.27. A function $f : [a, b] \to \mathbb{R}$ is said to be Lebesgue integrable provided that both f and $|f|$ are Henstock-Kurzweil integrable.")

[2]See "An Introduction to The Gauge Integral also known as the generalized Riemann integral, the Henstock integral, the Kurzweil integral, the Henstock-Kurzweil integral, the HK-integral, the Denjoy-Perron integral, etc." by Eric Schechter.

[3]Different versions of the gauge integral on $\mathbb{R}^n$ are invented in order to ensure rotation invariance, iterated integral ("Fubini") and integrability of the divergence of every differentiable function. However, no version enjoys all these properties. See "MathOverflow:What are the obstructions for a Henstock-Kurzweil integral in more than one dimension?".

[4]A nice space of bad functions is more useful than a bad space of nice functions; see "MathStackExchange:Why are gauge integrals not more popular?". Lebesgue integration leads to the Hilbert space $L_2(\mathbb{R}^n)$ and Banach spaces $L_p(\mathbb{R}^n)$; gauge integration fails to do so (but can be used as another entry to Lebesgue's theory). In addition, Lebesgue integration works in spaces much more general than $\mathbb{R}^n$.

[5]I mean, definable without parameters.

[6]Clearly, $m^*(G) = v_*(G)$.

Proof. We take a box B such that $G \subset B^\circ$, then $v_*(G) = {}_*\int_{\mathbb{R}^n} 1\!\!1_G = {}_*\int_B 1\!\!1_G = \sup_P L(1\!\!1_G, P)$, and $L(1\!\!1_G, P) = \sum_{C \in P} \operatorname{vol}(C) \inf_{\overline{C}} 1\!\!1_G = \sum_{C \in P, \overline{C} \subset G} \operatorname{vol}(C)$. By 6g7, $L(1\!\!1_G, P) = v(K_P)$ where $K_P = \cup_{C \in P, \overline{C} \subset G} \overline{C}$ is a compact subset of G. Thus, $v_*(G) = \sup_P v(K_P) \le \sup_{K \subset G} v^*(K)$. And "$\ge$" follows from (8e3). $\square$

8e7 Exercise. Every compact set is Lebesgue measurable. That is,

$$v^*(K) = \inf_{G \supset K} v_*(G) \quad \text{for every compact } K \subset \mathbb{R}^n\,,$$

the infimum being taken over all open bounded $G \supset K$.

Prove it.[1]

We see that

$$m^*(A) \le m(\overline{A}) = v^*(\overline{A}) = v^*(A) \quad \text{for all bounded sets } A \subset \mathbb{R}^n\,. \tag{8e8}$$

Given sets $X, X_1, X_2, \dots$ we write $X_i \uparrow X$ when $X_1 \subset X_2 \subset \dots$ and $\cup_i X_i = X$. Similarly, we write $X_i \downarrow X$ when $X_1 \supset X_2 \supset \dots$ and $\cap_i X_i = X$.

8e9 Proposition. *(Monotone convergence for open sets)* For all open bounded sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,

$$G_i \uparrow G \quad \Longrightarrow \quad v_*(G_i) \uparrow v_*(G)\,.$$

Proof. If $K \subset G$ is compact, then $\exists i \;\; K \subset G_i$, therefore $v^*(K) \le v_*(G_i) \le \lim_i v_*(G_i)$. By 8e6, supremum over K gives $v_*(G) \le \lim_i v_*(G_i)$. And "$\ge$" is evident. $\square$

8e10 Corollary. $v_*(G_1 \cup G_2 \cup \dots) \le v_*(G_1) + v_*(G_2) + \dots$ for all open $G_1, G_2, \dots \subset \mathbb{R}^n$ whose union is bounded.

Proof. $v_*(G_1 \cup \dots \cup G_i) \le v_*(G_1) + \dots + v_*(G_i)$ by (8e4), and $G_1 \cup \dots \cup G_i \uparrow G_1 \cup G_2 \cup \dots$; use 8e9. $\square$

8e11 Exercise. *(Monotone convergence for compact sets)* For all compact sets $K, K_1, K_2, \dots \subset \mathbb{R}^n$,

$$K_i \downarrow K \quad \Longrightarrow \quad v^*(K_i) \downarrow v^*(K)\,.$$

Prove it.

8e12 Exercise. It can happen that $K_i \uparrow A$ but $v^*(K_i) \uparrow b < v^*(A)$. Find an example. What about $G_i \downarrow A$?

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[1] Hint: apply 8e6 to $B^\circ \setminus K$.

8f Lebesgue's criterion for Riemann integrability

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. One says that f is continuous *almost everywhere,* if the set A of all discontinuity points of f satisfies $m^*(A) = 0$; or equivalently, is Lebesgue measurable, of Lebesgue measure zero. More generally, a property of a point of $\mathbb{R}^n$ is said to hold almost everywhere if it holds outside a set of Lebesgue measure zero.

8f1 Theorem. A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if it is continuous almost everywhere.

By (8e8), every (bounded) set of volume zero is of Lebesgue measure zero.

8f2 Lemma. Let $A_1, A_2, \dots \subset \mathbb{R}^n$ and $A = A_1 \cup A_2 \cup \dots$ be bounded. If all A_i are of Lebesgue measure zero then A is of Lebesgue measure zero.

Proof. Given $\varepsilon > 0$, we take open sets $G_i \supset A_i$ such that $v_*(G_i) \le \varepsilon/2^i$ and the open set $G = G_1 \cup G_2 \cup \dots$ is bounded; then $G \supset A$ and $v_*(G) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \dots = \varepsilon$ by 8e10. $\square$

Therefore $m(A) = 0$ whenever A_i are of volume zero. In particular, $m(A) = 0$ whenever A is countable (even if A is dense in a box, in which case $m(A) < m(\overline{A}) = v^*(\overline{A}) = v^*(A)$).

8f3 Lemma. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. If f is negligible then $f(\cdot) = 0$ almost everywhere.[1]

Proof. We consider sets $A = \{x : f(x) \ne 0\}$ and $A_i = \{x : |f(x)| \ge \frac{1}{i}\}$; $A = \cup_i A_i$. For each i we have $1\!\!1_{A_i} \le i|f|$, thus $v^*(A_i) \le i \,{}^*\!\!\int |f| = 0$, which implies $m(A_i) = 0$ and, by 8f2, $m(A) = 0$. $\square$

8f4 Lemma. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. If $|f|$ is upper semicontinuous and $f(\cdot) = 0$ almost everywhere, then f is negligible.

Proof. By semicontinuity, the sets A_i are closed (think, why), and compact. Therefore, $v^*(A_i) = m(A_i) \le m(A) = 0$. We have $|f| \le \frac{1}{i} 1\!\!1_A$ outside a set of volume zero; therefore ${}^*\!\!\int |f| \le \frac{1}{i} v^*(A)$ for all i. $\square$

8f5 Corollary. A bounded function f with bounded support is negligible if and only if the upper envelope of $|f|$ equals zero almost everywhere.

***Proof** of Theorem 8f1.* By 8d3, f is integrable if and only if Osc_f is negligible. We apply 8f5 to Osc_f and note that $|\operatorname{Osc}_f|^* = (\operatorname{Osc}_f)^* = \operatorname{Osc}_f$, since $\operatorname{Osc}_f = f^* - f_*$ is nonnegative and upper semicontinuous. $\square$

[1] The converse fails; try indicator of a dense countable set.

8f6 Exercise. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be bounded functions with bounded support. If $\operatorname{mid}(-M, f, M)$ and $\operatorname{mid}(-M, g, M)$ are integrable for all $M > 0$, then $\operatorname{mid}(-M, f + g, M)$ also is.[1]

Prove it.

8f7 Exercise. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be bounded functions with bounded support, $X = \{(f(x), g(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^2$, and $\varphi : X \to \mathbb{R}$ a continuous function, $\varphi(0, 0) = 0$. Then the function $\varphi(f(\cdot), g(\cdot))$ is integrable.[2]

Prove it.

[1] For "mid" see Sect. 8a.

[2] It is easy to find an elementary proof assuming that φ is continuous on the closure of X. However, X need not be closed, and φ need not be uniformly continuous.