

A Structures and isomorphisms in general¹

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This appendix is not a part of the course. Read it if you like to understand various kinds of spaces mentioned in Sect. 1 (vector, affine, topological, metric, metrizable, normed, Euclidean vector, Euclidean affine, Cartesian, classical Euclidean) as special cases of a general notion “mathematical structure” that embraces also groups, rings and other algebraic structures, as well as partially ordered sets, graphs etc.

A1 A simple example of structure: natural numbers²

Natural numbers may be defined as a set $\mathbb{N} = \{0, 1, 2, \dots\}$ endowed with a map $S : \mathbb{N} \rightarrow \mathbb{N}$ (so-called successor function) and a special element $0 \in \mathbb{N}$ satisfying appropriate conditions (so-called Peano axioms). Then, by definition, $1 = S(0)$, $2 = S(1) = S(S(0))$, and so on.

The theory of sets provides a well-known implementation of natural numbers:³

$$0 = \emptyset, \quad 1 = \{0\} = \{\emptyset\}, \quad 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \dots; \quad S(n) = n \cup \{n\}.$$

¹Probably, the general theory of structures may be called “abstract nonsense” similarly to this: ‘...category theory is the study of the general form of mathematical theories, without regard to their content. As a result, a proof that relies on category theoretic ideas often seems slightly out of context to those who are not used to such abstraction ... Such proofs are sometimes dubbed “abstract nonsense” ... A reader expecting a long, difficult proof might be surprised—or even delighted—by this bit of general nonsense.’ Wikipedia:Abstract nonsense.

²For detailed explanations about natural numbers see Chapters 9, 10 in a book: “The anatomy of mathematics” (second edition, 1974) by R.B. Kershner, L.R. Wilcox. Also, pages 30–33 in a book “Logical foundations of mathematics and computational complexity. A gentle introduction” (2013) by P. Pudlák.

³By finite ordinals.

Another, less well-known implementation:

$$0 = \emptyset, \quad 1 = \{0\} = \{\emptyset\}, \quad 2 = \{1\} = \{\{\emptyset\}\}, \dots; \quad S(n) = \{n\}.$$

These two implementations $(\mathbb{N}_1, 0_1, S_1)$, $(\mathbb{N}_2, 0_2, S_2)$ are isomorphic; a map $\varphi : \mathbb{N}_1 \rightarrow \mathbb{N}_2$ defined recursively by

$$(A1a) \quad \varphi(0_1) = 0_2, \quad \forall n \in \mathbb{N}_1 \quad \varphi(S_1(n)) = S_2(\varphi(n))$$

is an isomorphism (to be defined soon). The same holds for two arbitrary implementations. Note that, say, $\varphi(5_1) = 5_2$.

Addition is introduced recursively:

$$(A1b) \quad \forall n \in \mathbb{N} \quad n + 0 = n; \quad \forall m, n \in \mathbb{N} \quad m + S(n) = S(m + n).$$

Doing so in $(\mathbb{N}_1, 0_1, S_1)$, $(\mathbb{N}_2, 0_2, S_2)$ we get $+_1, +_2$ and observe that φ is also an isomorphism of semigroups $(\mathbb{N}_1, +_1)$, $(\mathbb{N}_2, +_2)$. That is,

$$\forall m, n \in \mathbb{N} \quad \varphi(m + n) = \varphi(m) + \varphi(n).$$

A2 Structures in general¹

Mathematics [...] cannot be explained completely by a single concept such as the mathematical structure. Nevertheless, Bourbaki's structuralist approach is the best that we have. ²

Evident as the notion of mathematical structure may seem these days, it was at least not made explicit until the middle of the 20th century. Then it was the influence of the Bourbaki-project and then later the development of category theory which made the notion explicit. ³

Given an arbitrary set X , one may produce many other sets, such as the Cartesian product $X \times X$, the set of all subsets of X denoted by $\mathcal{P}(X)$, and even such monster as (for instance)

$$\mathcal{P}(\mathcal{P}(X \times X) \times X \times \mathcal{P}(\mathcal{P}(X))) \times X.$$

¹Following the textbook by P. Pudlák (see Chapter 1), a monograph “Elements of mathematics: Theory of sets” (1968) by N. Bourbaki (English translation) (see Chapter IV, and §8 in “Summary of results”), and an article “Sentences of type theory: the only sentences preserved under isomorphisms” by M.V. Marshall, R. Chuaqui (1991), The Journal of Symbolic Logic **56**:3, 932–948 (see §2).

²Pudlák, page 3.

³nLab:Structure#related entries.

The latter may be obtained substituting by X the indeterminate, denoted “ \square ”, in the formula

$$F = \text{“}\mathcal{P}(\mathcal{P}(\square \times \square) \times \square \times \mathcal{P}(\mathcal{P}(\square))) \times \square\text{”} .$$

This formula is an element of a formal language L in the alphabet of 5 characters “ \square ”, “ \times ”, “ \mathcal{P} ”, “(”, “)” ; we define L as the least set of words satisfying

$$\begin{aligned} \text{“}\square\text{”} &\in L; \\ \forall F, G \in L \quad F \text{ “}\times\text{”} G &\in L; \\ \forall F \in L \quad \text{“}\mathcal{P}(\text{“} F \text{“})\text{”} &\in L. \end{aligned}$$

From now on, formulas of L will be called *types* (for conforming with the terminology of the general theory of structures). We define $F(X)$ for a type F and a set X recursively:

$$F(X) = \begin{cases} X & \text{if } F = \text{“}\square\text{”}; \\ G(X) \times H(X) & \text{if } F = G \text{ “}\times\text{”} H; \\ \mathcal{P}(G(X)) & \text{if } F = \text{“}\mathcal{P}(\text{“} G \text{“})\text{”}. \end{cases}$$

A *structure* of type F on X is, by definition, an element of $F(X)$.

Thus,

- * every element of X is a structure on X ;
- * every pair of structures on X is a structure on X ;
- * every set of structures on X of the same type is a structure on X ;

and nothing else is a structure on X .

A2a Example. If $(\mathbb{N}, 0, S)$ is an implementation of natural numbers then $(0, S)$ is a structure on \mathbb{N} of type¹ $\square \times \mathcal{P}(\square \times \square)$, since a function is a subset of the Cartesian product. Introducing addition we get on \mathbb{N} a structure $(0, S, +)$ of type $\square \times \mathcal{P}(\square \times \square) \times \mathcal{P}(\square \times \square \times \square)$. Introducing also order “ \leq ” by $m \leq n \iff \exists k \ m + k = n$ we get another structure $(0, S, +, \leq)$ of type $\square \times \mathcal{P}(\square \times \square) \times \mathcal{P}(\square \times \square \times \square) \times \mathcal{P}(\square \times \square)$.

Algebraic structures do not need $\mathcal{P}(\mathcal{P}(\square))$, but non-algebraic structures do. Most important, a *topology* on a set X is usually defined as a set τ of subsets of X (called “open sets”) satisfying appropriate conditions (see Sect. 1c). Thus, $\tau \subset \mathcal{P}(X)$, that is, $\tau \in \mathcal{P}(\mathcal{P}(X))$ is a structure of type $\mathcal{P}(\mathcal{P}(\square))$.

¹From now on we do not bother to put types into quotes.

In full generality, types like $\mathcal{P}(\mathcal{P}(\mathcal{P}(\square)))$ (and more complicated) are allowed. But in practice they are used rarely (maybe never).

Some structures need a so-called *auxiliary base set* E in addition to a *principal base set* X . Most important, the structure of a vector space on X stipulates not only addition $X \times X \rightarrow X$ but also multiplication by scalar $\mathbb{R} \times X \rightarrow X$. One adds a new character, “ E ”, to the alphabet; a new line, “ E ” $\in L$, to the definition of the language L ; defines $F(X, E)$ for arbitrary sets X, E ; and uses $F(X, \mathbb{R})$ (or another field, as needed, instead of \mathbb{R}).

Some structures need two principal base sets. For example, graphs (and multigraphs) are defined in terms of vertices and edges. Vertices are a principal base set. One approach (“edges without own identity”) treats edges as (ordered or unordered) pairs of vertices. The other approach (“edges with own identity”) treats edges as the second principal base set.

In full generality, any finite number of principal base sets and any finite number of auxiliary base sets are allowed. But in practice, more than two base sets are used rarely (maybe never).

A3 Transport of structures; isomorphism¹

Given two sets X_1, X_2 and a bijection $\varphi : X_1 \rightarrow X_2$, we define the *canonical extension*

$$F(\varphi) : F(X_1) \rightarrow F(X_2)$$

for every type F as follows.

For $F = \square$ we have $F(X_1) = X_1$, $F(X_2) = X_2$, and take $F(\varphi) = \varphi$, that is, $F(\varphi)(x) = \varphi(x)$ for $x \in X_1$.

For $F = \square \times \square$ we have $F(X_1) = X_1 \times X_1$, $F(X_2) = X_2 \times X_2$, and take $F(\varphi)(x, y) = (\varphi(x), \varphi(y))$ for $x, y \in X_1$.

For $F = \mathcal{P}(\square)$ we have $F(X_1) = \mathcal{P}(X_1)$, $F(X_2) = \mathcal{P}(X_2)$, and take $F(\varphi)(A) = \{\varphi(x) : x \in A\}$ (the image) for $A \subset X_1$.

Further, we continue the process recursively.

If $F = G \times H$, that is, a structure of type F is a pair of structures (of types G, H), then we have $F(X_1) = G(X_1) \times H(X_1)$, $F(X_2) = G(X_2) \times H(X_2)$, and take $F(\varphi)(\sigma, \tau) = (G(\varphi)(\sigma), H(\varphi)(\tau))$ for $\sigma \in G(X_1)$, $\tau \in H(X_1)$.

Finally, if $F = \mathcal{P}(G)$, that is, a structure of type F is a set of structures of type G , then we have $F(X_1) = \mathcal{P}(G(X_1))$, $F(X_2) = \mathcal{P}(G(X_2))$, and take $F(\varphi)(A) = \{G(\varphi)(\sigma) : \sigma \in A\}$ (the image) for $A \subset G(X_1)$.

¹For a lightweight introduction to isomorphism see Chapter 16 in “The anatomy of mathematics”.

A3a Definition. Let F be a type, X_1, X_2 sets, σ_1 a structure of type F on X_1 , and σ_2 a structure of type F on X_2 . A bijection $\varphi : X_1 \rightarrow X_2$ is an *isomorphism* between (X_1, σ_1) and (X_2, σ_2) if $F(\varphi)(\sigma_1) = \sigma_2$.

A3b Exercise. Prove that (A1a) treated as a condition on φ is necessary and sufficient for φ to be an isomorphism.

A3c Exercise. Define a partially ordered set as a set endowed with a structure of the type $\square \times \square$, and prove that φ is an isomorphism between two partially ordered sets if and only if

$$\forall x, y \ (x \leq y \iff \varphi(x) \leq \varphi(y)) .$$

A3d Exercise. Define a group as a set endowed with a structure, and prove that the two notions of isomorphism agree (one defined in group theory, the other defined here).

As we know, some structures need an auxiliary base set E . Transport of such structures leaves intact elements of E (by definition). For example, an isomorphism $\varphi : X_1 \rightarrow X_2$ of vector spaces X_1, X_2 ¹ satisfies $\forall x \in X_1 \ \forall \lambda \in \mathbb{R} \ \varphi(\lambda x) = \lambda \varphi(x)$, not $\varphi(\lambda x) = \varphi(\lambda) \varphi(x)$, nor $\varphi(\lambda x) = \psi(\lambda) \varphi(x)$.

A long list of special cases of the general notion of isomorphism is available on Wikipedia.²

A3e Exercise.³ For three sets X_1, X_2, X_3 and two bijections $\varphi : X_1 \rightarrow X_2$, $\psi : X_2 \rightarrow X_3$ prove that $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ for every type F .

Also, $F(\varphi^{-1}) = (F(\varphi))^{-1}$, and $F(\text{id}) = \text{id}$ (for $X_1 = X_2$).

In the special case of $X_1 = X_2 (= X)$ and $\sigma_1 = \sigma_2 (= \sigma)$ isomorphisms are called *automorphisms* (or symmetries) of (X, σ) . It follows from A3e that for each σ the automorphisms of (X, σ) are a subgroup of the group of all bijections $X \rightarrow X$.

A4 Transportable properties

A type gives only preliminary information on a structure. Main information is given by so-called axioms (of structure).

¹Are you disturbed by this formulation? This is “abuse of language”, see Sect. 1b (p. 9).

²Wikipedia:Equivalent definitions of mathematical structures#Transport of structures; isomorphism

³If you are acquainted with categories and functors, observe that F is a functor from the category **Set*** of sets and bijections to itself.

A4a Example. Peano axioms for natural numbers:

$$\begin{aligned} & \forall n \in \mathbb{N} \quad S(n) \neq 0; \\ & \forall m, n \in \mathbb{N} \quad (S(m) = S(n) \implies m = n); \\ & \forall K \in \mathcal{P}(\mathbb{N}) \quad ((0 \in K \wedge \forall n (n \in K \implies S(n) \in K)) \implies K = \mathbb{N}). \end{aligned}$$

The third axiom, of induction, prevents \mathbb{N} from containing elements beyond $0, S(0), S(S(0)), \dots$

Given $(\mathbb{N}_1, 0_1, S_1)$, \mathbb{N}_2 and a bijection $\varphi : \mathbb{N}_1 \rightarrow \mathbb{N}_2$, we consider $(\mathbb{N}_2, 0_2, S_2)$ where $(0_2, S_2) = F(\varphi)(0_1, S_1)$, $F = \square \times \mathcal{P}(\square \times \square)$, and observe that Peano axioms are *transportable* in the following sense:

if $(\mathbb{N}_1, 0_1, S_1)$ satisfies the axioms then $(\mathbb{N}_2, 0_2, S_2)$ satisfies the axioms.

Here is a proof that the first Peano axiom is transportable. Given $n_2 \in \mathbb{N}_2$, we consider $n_1 = \varphi^{-1}(n_2) \in \mathbb{N}_1$. By A3b, $\varphi(S_1(n_1)) = S_2(n_2)$ and $\varphi(0_1) = 0_2$. We know that $S_1(n_1) \neq 0_1$. It follows that $\varphi(S_1(n_1)) \neq \varphi(0_1)$, that is, $S_2(n_2) \neq 0_2$.

A4b Exercise. Prove transportability of the other two Peano axioms.

Here is an example of a non-transportable property of $(0, S)$:

$$S(S(0)) = \{\{\emptyset\}\}.$$

Another example:

$$0 \in S(S(0)).$$

Clearly, these are not transportable because they involve objects ($\{\{\emptyset\}\}$) or relations (\in) beyond the given structure. This is sometimes called “evil”.¹

Such “evil” never appears in mathematical practice, since a mathematician always treats elements of a principal base set as points with no internal structure. In the absence of “evil”, transportability is not an issue; it always holds, and is easy to prove. If you proved it few times, you can prove it always.²

¹“For a category theorist, making a distinction between one-element sets is *evil*. Instead of looking inside an object to see how its made, we should only care about how it interacts with the world around it.” (Mike Stay, “Cartesian categories and the problem of evil”) “Floating around the web (and maybe the nLab) is the idea of half-jokingly referring to a breaking of equivalence invariance as evil. This is probably meant as a pedagogical way of amplifying that it is to be avoided.” (nLab:principle of equivalence#terminology)

²A model-theoretic criterion of transportability is available, see Marshall and Chuaqui.

More formally, given a property T of structures of a type F , we introduce a set

$$T(X) \subset F(X)$$

of all structures that have the property T . The property T is called transportable, if

$$\sigma_1 \in T(X_1) \iff \sigma_2 \in T(X_2)$$

whenever X_1, X_2 are sets, $\varphi : X_1 \rightarrow X_2$ is a bijection, $\sigma_1 \in F(X_1)$, $\sigma_2 \in F(X_2)$, and $F(\varphi) : \sigma_1 \mapsto \sigma_2$.

A4c Exercise. A semigroup is, by definition, a set X endowed with a binary operation $(x, y) \mapsto xy$ that is associative: $\forall x, y, z \in X \ (xy)z = x(yz)$. Prove that associativity is transportable.

A4d Exercise. Axioms for topology on X :¹

\emptyset and X are open;

intersection of two (or finitely many) open sets is open;

union of open sets (no matter how many) is open.

Prove transportability of these axioms.

A4e Exercise. A topological space² is called compact, if every open covering contains a finite subcovering. Prove transportability of compactness.

A5 Species of structures

Given a transportable property T of structures of a type F , two sets X_1, X_2 and a bijection $\varphi : X_1 \rightarrow X_2$, we introduce the restricted transport map $T(\varphi) = F(\varphi)|_{T(X_1)}$ and note that³

$$T(\varphi) : T(X_1) \rightarrow T(X_2) \text{ bijectively.}$$

Such T is called a *species* of structures (of the type F).

A5a Example. Peano axioms lead to a species of the type $\square \times \mathcal{P}(\square \times \square)$; let us call it the Peano species.⁴ In this case $T(X) \neq \emptyset$ if and only if X is countably infinite.

A5b Example. Semigroups are a species of the type $\mathcal{P}(\square \times \square \times \square)$.

A5c Corollary. Let $\varphi : X_1 \rightarrow X_2$ be a bijection, $\sigma_1 \in T(X_1)$ and $\sigma_2 \in T(X_2)$. According to A3a, φ is an isomorphism between (X_1, σ_1) and (X_2, σ_2) if and only if $T(\varphi)(\sigma_1) = \sigma_2$.

¹See Sect. 1c.

²See Sect. 1c.

³A functor, again.

⁴Not a standard terminology.

A6 Relations between species of structures

Taking into account that every pair of structures on X is a structure on X we may use a transportable property of the pair as an axiom of this combined structure.

A6a Example. Let $(\mathbb{N}, 0, S)$ be a set endowed with a Peano structure,¹ and $(\mathbb{N}, +)$ a semigroup. Then $(\mathbb{N}, 0, S, +)$ is a set endowed with both structures, and relations (A1b) (evidently transportable!) may be treated as axioms of this combined species.

Generally, if F_1, F_2 are two types then $F = F_1 \times F_2$ is also a type. Given a species T_1 of structures of the type F_1 and a species T_2 of structures of the type F_2 , we get a species $T_1 \times T_2$ of structures of the type F ,

$$(T_1 \times T_2)(X) = T_1(X) \times T_2(X).$$

Additional axioms may lead to a smaller species T of structures of the type F ,

$$T(X) \subset T_1(X) \times T_2(X),$$

and $T(X)$ may be thought of as a binary relation. Every relation T between structures may be treated as a property of the combined structure, and we wonder, is it transportable? If it is, then T is also a species.

A6b Exercise. Which of the relations listed below are transportable?

- | | |
|---|--|
| (a) $a = b$ for $a, b \in X$; | (b) $a = b$ for $a \in X, b \in \mathcal{P}(X)$; |
| (c) $a = b$ for $a, b \in \mathcal{P}(X)$; | (d) $a \in b$ for $a, b \in X$; |
| (e) $a \in b$ for $a \in X, b \in \mathcal{P}(X)$; | (f) $a \in b$ for $a, b \in \mathcal{P}(X)$; |
| (g) $a \subset b$ for $a, b \in X$; | (h) $a \subset b$ for $a \in X, b \in \mathcal{P}(X)$; |
| (i) $a \subset b$ for $a, b \in \mathcal{P}(X)$; | (j) $a \cup b = c$ for $a, b, c \in X$; |
| (k) $a \cup b = c$ for $a, b, c \in \mathcal{P}(X)$; | (l) $\cup_{z \in a} z = b$ for $a \in \mathcal{P}(\mathcal{P}(X)), b \in \mathcal{P}(X)$. |

A transportable binary relation $T(X) \subset T_1(X) \times T_2(X)$ may appear to be a function from $T_1(X)$ to $T_2(X)$; this happens if and only if

$$\forall \sigma_1 \in T_1(X) \exists! \sigma_2 \in T_2(X) \quad (\sigma_1, \sigma_2) \in T(X).$$

In this case we write

$$T(X) : T_1(X) \rightarrow T_2(X).$$

If this is the case for all X , we write

$$T : T_1 \rightarrow T_2$$

¹That is, a structure of the Peano species.

and say that T is a *procedure of deduction* of a structure of the species T_2 from a structure of the species T_1 .

It may happen that the converse holds as well:

$$\forall \sigma_2 \in T_2(X) \exists! \sigma_1 \in T_1(X) \quad (\sigma_1, \sigma_2) \in T(X)$$

for all X . Such T is an *equivalence* between T_1 and T_2 . Informally it means that T_1 and T_2 describe the same structure in different forms.

If T is not an equivalence, still, it may happen that the mapping $T(X) : T_1(X) \rightarrow T_2(X)$ is injective (that is, one-to-one) for all X . In this case we say that T *remembers structure* and *forgets properties*.¹ Similarly, it may happen that $T(X)$ is surjective (that is, onto) for all X . In this case we say that T *remembers properties* and *forgets structure*. Informally it means that every $\sigma_2 \in T_2(X)$ may be *upgraded* to $\sigma_1 \in T_1(X)$, but not uniquely; and σ_1 is *downgraded* to σ_2 by T . Of course, T may forget both structure and properties; on the other hand, an equivalence remembers everything.

In particular, A6a provides a procedure of deduction of a semigroup structure from a Peano structure.

A6c Exercise. Denoting by T the procedure given by A6a prove that the mapping $T(X) : T_1(X) \rightarrow T_2(X)$ is one-to-one.²

A6d Exercise. Invent a procedure of deduction of a group structure from a vector space structure.

A6e Exercise.³ For arbitrary procedure of deduction $T : T_1 \rightarrow T_2$, sets X_1, X_2 and bijection $\varphi : X_1 \rightarrow X_2$ the diagram

$$\begin{array}{ccc} T_1(X_1) & \xrightarrow{T_1(\varphi)} & T_1(X_2) \\ T(X_1) \downarrow & & \downarrow T(X_2) \\ T_2(X_1) & \xrightarrow{T_2(\varphi)} & T_2(X_2) \end{array}$$

is commutative, that is, $T_2(\varphi) \circ T(X_1) = T(X_2) \circ T_1(\varphi)$.

Prove it.⁴

¹In the spirit of nLab:Stuff, structure, property.

²Hint: it is easy to characterize the number $0 \in \mathbb{N}$ in terms of the addition on \mathbb{N} , and a bit harder to characterize the number $1 \in \mathbb{N}$.

³If you are acquainted with categories and functors, observe that T is a natural transformation from the functor T_1 to the functor T_2 .

⁴Hint: recall the definition of $F(\varphi)$ for $F = F_1 \times F_2$.

A7 General isomorphism theorems¹

Every isomorphism between sets endowed with Peano structures is also an isomorphism between the corresponding semigroups. Every isomorphism between vector spaces is also an isomorphism between the corresponding groups. These facts are special cases of the following general fact.

A7a Theorem. (*The first general isomorphism theorem*)²

Let $T : T_1 \rightarrow T_2$ be a procedure of deduction, X_1, X_2 sets, $\sigma_1 \in T_1(X_1)$, $\sigma_2 \in T_1(X_2)$, and $\varphi : X_1 \rightarrow X_2$ a bijection. If φ is an isomorphism between (X_1, σ_1) and (X_2, σ_2) then φ is an isomorphism between $(X_1, T(X_1)(\sigma_1))$ and $(X_2, T(X_2)(\sigma_2))$.

Proof. By A5c, $T_1(\varphi)(\sigma_1) = \sigma_2$; we have to prove that $T_2(\varphi)(T(X_1)(\sigma_1)) = T(X_2)(\sigma_2)$, that is, $T_2(\varphi) \circ T(X_1)(\sigma_1) = T(X_2) \circ T_1(\varphi)(\sigma_1)$. This equality follows from A6e:

$$\begin{array}{ccc} \sigma_1 & \xrightarrow{T_1(\varphi)} & \sigma_2 \\ T(X_1) \downarrow & & \downarrow T(X_2) \\ T(X_1)(\sigma_1) & \xrightarrow{T_2(\varphi)} & T(X_2)(\sigma_2) \end{array}$$

□

Every isomorphism between vector spaces sends a basis into a basis, a subspace into a subspace, etc. These facts are special cases of the following general fact.

A7b Theorem. (*The second general isomorphism theorem*)³

Let $T : T_2 \rightarrow T_1$ be a procedure of deduction, X_1, X_2 sets, $\sigma_1 \in T_1(X_1)$, $\sigma_2 \in T_1(X_2)$, and $\varphi : X_1 \rightarrow X_2$ a bijection. If φ is an isomorphism between (X_1, σ_1) and (X_2, σ_2) , $\tau_1 \in T_2(X_1)$ satisfies $T(X_1)(\tau_1) = \sigma_1$, and $\tau_2 = T_2(\varphi)(\tau_1)$ then $T(\tau_2) = \sigma_2$.

A7c Exercise. Prove the theorem above.⁴

A7d Example. Consider the species T_1 of vector spaces (over \mathbb{R}), the species T_2 of vector spaces endowed with a (“chosen”, “preferred”) basis, and the

¹Probably, general isomorphism theorems may be called “profound triviality” similarly to this: “The Baire category is a profound triviality which condenses the folk wisdom of a generation of ingenious mathematicians into a single statement.” (T.W. Körner, “Linear analysis” Sect.6, p.13.)

²Not a standard terminology.

³Not a standard terminology.

⁴Draw a diagram.

canonical deduction procedure¹ $T : T_2 \rightarrow T_1$. Let φ be an isomorphism between vector spaces X_1, X_2 . By Theorem A7b, if (e_1, \dots, e_n) is a basis of X_1 then $(\varphi(e_1), \dots, \varphi(e_n))$ is a basis of X_2 .

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¹The so-called forgetful functor.