Some solutions

Question 1

Assuming that the given extremum is a maximum, we have a neighborhood $U \subset \mathbb{R}^n$ of x_0 such that

$$\forall x \in Z \cap U \;\; f(x) \le f(x_0) \, .$$

Similarly to the proof of Theorem 3a1, it follows from 3a2 that the mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^{16}$, defined by $\varphi(x) = \big(\varphi_1(x), \dots, \varphi_{16}(x) \big)$, is open at x_0. Thus (according to 2a6(b)) we have a neighborhood $V \subset \mathbb{R}^{16}$ of the point $y_0 = \varphi(x_0)$ such that $\varphi(U) \supset V$. WLOG, $V = V_1 \times V_2$ where $V_1 \subset \mathbb{R}^{10}$ is a neighborhood of $\big(\varphi_1(x_0), \dots, \varphi_{10}(x_0) \big) = 0$ and $V_2 \subset \mathbb{R}^6$ is a neighborhood of $\big(\varphi_{11}(x_0), \dots, \varphi_{16}(x_0) \big)$.

For every $y \in V_2$ there exists $x \in U$ such that $\varphi(x) = (0, y)$, that is, $\big(\varphi_1(x), \dots, \varphi_{10}(x) \big) = 0$ and $\big(\varphi_{11}(x), \dots, \varphi_{16}(x) \big) = y$. We have $x \in Z$ and $f(x) = g(y)$. Taking into account that $f(x) \le f(x_0)$ we get

$$g(y) \le f(x_0) \quad \text{for all } y \in V_2 \, .$$

For every $x \in \mathbb{R}^n$ close enough to x_0 we have $\varphi(x) \in V$, that is, $\varphi(x) = (u, y)$ where $u \in V_1$ and $y \in V_2$. Thus, $f(x) = g(y) \le f(x_0)$. □

Question 2

One solution

For every box $B = [a, b] \times [c, d]$, its image $\varphi(B) = \{ (x, y) : a \le x \le b, c + f(x) \le y \le d + f(x) \}$ is admissible; therefore the set

$$E = \{ (x, y) : a \le x \le b, 0 \le y \le d + f(x) \} = \varphi(B) \cap ([a, b] \times [0, d + M])$$

is also admissible (being the intersection of two admissible sets), provided that $c \le -M$ and $d \ge M$ where $M = \sup_x |f(x)|$.

It follows that[1]

$$\int_a^b v_1(E_x) \, \mathrm{d}x = v_2(E)$$

by (5e3), which also ensures integrability of the function $x \mapsto v_1(E_x) = d + f(x)$ on $[a, b]$. Being integrable on every interval, f is continuous almost everywhere by 6d2 (and 6c2). □

[1] A similar argument was used in 5e7(a).

Another solution

For every box $B = [a, b] \times [c, d]$, its image $\varphi(B) = \{(x, y) : a \le x \le b, c + f(x) \le y \le d + f(x)\}$ is admissible; therefore its boundary $E = \partial(\varphi(B))$ has area 0 by 6b8(b).

It follows that

$$\int_a^b v_1(E_x)\,\mathrm{d}x = 0$$

by (5e3). We note that $v_1(E_x) = 2\operatorname{Osc}_f(x)$, since $E_x = [c + f_*(x), c + f^*(x)] \cup [d + f_*(x), d + f^*(x)]$, provided that $c < -M$ and $d > M$ where $M = \sup_x |f(x)|$. The equality $\int_a^b \operatorname{Osc}_f = 0$ shows that f is continuous almost everywhere on $[a, b]$ (and therefore on the whole $\mathbb{R}$). □

A different approach inspired by the work of a student

1. Using (5e3), prove that $v_2\big(\varphi(B)\big) = v_2(B)$ for every box $B \subset \mathbb{R}^2$.[1]

2. Using linear combination of indicators, prove that for every step function $g : \mathbb{R}^2 \to \mathbb{R}$ the function $g \circ \varphi^{-1}$ is integrable, and $\int g \circ \varphi^{-1} = \int g$.

3. Using sandwich, generalize it for all integrable g.[2]

4. Applying it to $g(x, y) = y$ (on a large box) we see that the function $(x, y) \mapsto y - f(x)$ is integrable on every box.

5. It follows that the function $(x, y) \mapsto f(x)$ is integrable on every box.

6. Using 5d1 prove that the function f is integrable on every box (and therefore, continuous almost everywhere). □

A brilliant solution (sketch) found by a student

For $B = [0, 1] \times [0, 1]$ we have such inequality on Darboux sums:

$$U_N(f) - L_N(f) \le U_N(\mathbb{1}_{\varphi(B)}) - L_N(\mathbb{1}_{\varphi(B)})\,.$$

(Draw a picture and see, why.) The claim follows easily! □

[1] Though, the same argument gives it for all admissible sets.

[2] Similar to 5d4.