Some solutions

Question 1

According to Th. 3f1, locally (near x_0), M can be given by $(n-k)$ constraints $g_1, \dots, g_{n-k}$ that satisfy the conditions of Th. 3a1. The latter theorem gives Lagrange multipliers $\lambda_1, \dots, \lambda_{n-k}$ such that $\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \dots + \lambda_{n-k} \nabla g_{n-k}(x_0)$. The function

$$g = \lambda_1 g_1 + \dots + \lambda_{n-k} g_{n-k}$$

does the job. □

Question 2 (for continuous f)

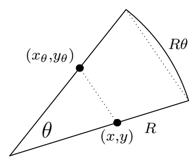
We take $R \in (0, \infty)$ such that $f(x,y) = 0$ whenever $|(x,y)| > R$, then also $f_\theta(x,y) = 0$ whenever $|(x,y)| > R$, since $f_\theta(x,y) = f(x_\theta, y_\theta)$ where $x_\theta = x\cos\theta - y\sin\theta$, $y_\theta = x\sin\theta + y\cos\theta$, and $|(x_\theta, y_\theta)| = |(x,y)|$.

We have $\int |f_\theta - f| \le \pi R^2 \max_{x,y} |f_\theta(x,y) - f(x,y)|$; thus, it is sufficient to get $|f_\theta(x,y) - f(x,y)| \le \varepsilon_1$ for all x, y; here $\varepsilon_1 = \frac{\varepsilon}{\pi R^2}$. That is, we need

$$\forall x, y \;\; |f(x_\theta, y_\theta) - f(x,y)| \le \varepsilon_1 \,.$$

Being continuous on the compact disk $|(x,y)| \le R$, the function f is uniformly continuous on this disk; thus, there exists $\delta_1 > 0$ such that $|f(x_\theta, y_\theta) - f(x,y)| \le \varepsilon_1$ whenever $|(x_\theta, y_\theta) - (x,y)| \le \delta_1$. Clearly,[1]

$$|(x_\theta, y_\theta) - (x,y)| \le R\theta \,,$$



therefore, the number

$$\delta = \frac{\delta_1}{R}$$

does the job. □

[1] In fact, $|(x_\theta, y_\theta) - (x,y)| \le 2R\sin\frac{1}{2}\theta$.