Topological notions in $\mathbb{R}^n$ are insensitive to a change of basis.
Topological notions are well-defined in every n-dimensional vector space, and preserved by isomorphisms of these spaces.

1f18 Exercise. (a) Determinant is a continuous function $A \mapsto \det A$ on $\mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$;
(b) invertible operators are an open set;
(c) the mapping $A \mapsto A^{-1}$ is continuous on this open set.

1f19 Exercise. If $A \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$ satisfies $\|A\| < 1$, then
(a) the series $\mathrm{id} - A + A^2 - A^3 + \dots$ converges in $\mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$;
(b) the sum S of this series satisfies $(\mathrm{id}+A)S = \mathrm{id}$, $S(\mathrm{id}+A) = \mathrm{id}$; thus, $\mathrm{id}+A$ is invertible;
(c) $\det(\mathrm{id}+A) > 0$.

When differentiating a given mapping, we may choose at will a pair of bases. This applies to any pair of finite-dimensional vector spaces.

1f23 Exercise. If $f \in C^1(U \to \mathbb{R}^m)$ and $g \in C^1(\mathbb{R}^m \to \mathbb{R}^\ell)$, then $g \circ f \in C^1(U \to \mathbb{R}^\ell)$.

1f24 Exercise. A mapping f is continuously differentiable if and only if all parial derivatives $D_i f_j$ exist and are continuous. (Here $f(x) = \big(f_1(x), \dots, f_m(x)\big)$.)

1f25 Exercise. (a) If $f \in C^1(U)$ and $g \in C^1(U \to \mathbb{R}^m)$, then $fg \in C^1(U \to \mathbb{R}^m)$ (pointwise product).
(b) If $f, g \in C^1(U \to \mathbb{R}^m)$, then $\langle f(\cdot), g(\cdot)\rangle \in C^1(U)$ (scalar product).

1f27 Exercise. (a) Determinant is a continuously differentiable function $f : A \mapsto \det A$ on $\mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$;
(b) $(Df)_{\mathrm{id}}(H) = \mathrm{tr}(H)$ for all $H \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$;
(c) $(D \log|f|)_A(H) = \mathrm{tr}(A^{-1}H)$ for all $H \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$ and all invertible $A \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$.

(1f31) $\quad |f(b) - f(a)| \le C|b-a|\,, \; C = \sup_{t \in (0,1)} \|(Df)_{a+t(b-a)}\|$ (finite increment theorem)

2a5 Exercise. For a linear $A : \mathbb{R}^n \to \mathbb{R}^m$ the following conditions are equivalent:
(a) A is invertible;
(b) A is a homeomorphism;
(c) A is a local homeomorphism;
(d) A is a diffeomorphism;
(e) A is a local diffeomorphism.

2a9 Exercise. For a linear $A : \mathbb{R}^n \to \mathbb{R}^m$ the following conditions are equivalent:
(a) $A(\mathbb{R}^n) = \mathbb{R}^m$ ("onto");
(b) A is open at 0;
(c) A is open.

2b1 Theorem (inverse function)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable near 0, $f(0) = 0$, and $(Df)_0 = A : \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Then f is a local diffeomorphism, and $\big(D(f^{-1})\big)_0 = A^{-1}$.

Similarly, $\big(D(f^{-1})\big)_{f(x)} = \big((Df)_x\big)^{-1}$ for all x near 0.

2b3 Theorem (implicit function)**.** Let $f : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable near $(0,0)$, $f(0,0) = 0$, and $(Df)_{(0,0)} = A = \big(B \,\big|\, C\big)$, $B : \mathbb{R}^{n-m} \to \mathbb{R}^m$, $C : \mathbb{R}^m \to \mathbb{R}^m$, with C invertible. Then there exists $g : \mathbb{R}^{n-m} \to \mathbb{R}^m$, continuously differentiable near 0, such that the two relations $f(x,y) = 0$ and $y = g(x)$ are equivalent for (x,y) near $(0,0)$; and $(Dg)_0 = -C^{-1}B$.

Similarly, $(Dg)_x = -C_x^{-1}B_x$, where $\big(B_x \,\big|\, C_x\big) = A_x = (Df)_{(x,g(x))}$, for all x near 0.

2c3 Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable near 0, $f(0) = 0$, and $(Df)_0 = A : \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Then f is open at 0.

2d5 Theorem. Let $f : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$ and A, B, C be as in Th. 2b3. Then there exists $g : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$, continuously differentiable near $(0,0)$, such that the two relations $f(x,y) = z$ and $y = g(x,z)$ are equivalent for (x,y,z) near $(0,0,0)$; and $(Dg)_{(0,0)} = \big(-C^{-1}B \,\big|\, C^{-1}\big)$.

3a1 Theorem (Lagrange multipliers)**.** Assume that $x_0 \in \mathbb{R}^n$, $1 \le m \le n-1$, functions $f, g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable near x_0, $g_1(x_0) = \dots = g_m(x_0) = 0$, and the vectors $\nabla g_1(x_0), \dots, \nabla g_m(x_0)$ are linearly independent. If x_0 is a local constrained extremum point of f subject to $g_1(\cdot) = \dots = g_m(\cdot) = 0$, then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $\quad \nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \dots + \lambda_m \nabla g_m(x_0)$.

$g_1(x) = \dots = g_m(x) = 0$	(m equations)	$\lambda_1, \dots, \lambda_m$	(m variables)
$\nabla f(x) = \lambda_1 \nabla g_1(x) + \dots + \lambda_m \nabla g_m(x)$	(n equations)	x	(n variables)

3a2 Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable near 0, $f(0) = 0$, and $(Df)_0 = A : \mathbb{R}^n \to \mathbb{R}^m$ be onto. Then f is open at 0.

$$M_p(x_1, \dots, x_n) = \left(\frac{x_1^p + \dots + x_n^p}{n}\right)^{1/p} \quad \text{for } x_k > 0\,; \quad M_p \le M_q \text{ for } p \le q\,.$$

The system of $m+n$ equations proposed in Sect. 3a is only one way of finding local constrained extrema. Not necessarily the simplest way.
No need to find ∇f when $f(\cdot) = \varphi(g(\cdot))$; find ∇g, note that ∇f is collinear to ∇g.
If Lagrange method does not solve a problem to the end, it may still give a useful information. Combine it with other methods as needed.

3d1 Proposition (singular value decomposition)**.** Every linear operator from one finite-dimensional Euclidean vector space to another sends some orthonormal basis of the first space into an orthogonal system in the second space.

3d2 Proposition. Every linear operator from an n-dimensional Euclidean vector space to an m-dimensional Euclidean vector space has a diagonal $m \times n$ matrix in some pair of orthonormal bases.

3d3 Proposition. Every finite-dimensional vector space endowed with two Euclidean metrics contains a basis orthonormal in the first metric and orthogonal in the second metric.

(3e1) $$\sup_{Z_c} f = \sup_{Z_0} f + \lambda_1(0)c_1 + \dots + \lambda_m(0)c_m + o(|c|)\,.$$

3f1 Theorem. The following conditions on a set $M \subset \mathbb{R}^n$, a point $x_0 \in M$ and a number $k \in \{1, 2, \dots, n-1\}$ are equivalent:

(a) there exists a mapping $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$, continuously differentiable near x_0, such that $(Df)_{x_0} = A : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is onto, and

$$x \in M \iff f(x) = f(x_0) \quad \text{for all } x \text{ near } x_0 ;$$

(b) there exists a local diffeomorphism φ near x_0 such that

$$x \in M \iff \varphi(x) \in \mathbb{R}^k \times \{0_{n-k}\} \quad \text{for all } x \text{ near } x_0 ;$$

(c) there exists a permutation $(i_1, \dots, i_n)$ of $\{1, \dots, n\}$ and a mapping $g : \mathbb{R}^k \to \mathbb{R}^{n-k}$, continuously differentiable near $(x_{0,i_1}, \dots, x_{0,i_k})$, such that

$$x \in M \iff g(x_{i_1}, \dots, x_{i_k}) = (x_{i_{k+1}}, \dots, x_{i_n}) \quad \text{for all } x \text{ near } x_0 .$$

A nonempty set $M \subset \mathbb{R}^n$ is a k-dimensional *manifold,* if the equivalent conditions 3f1(a,b,c) hold for every $x_0 \in M$.

3f3 Exercise. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism, and $M \subset \mathbb{R}^n$.

(a) If M is a k-manifold near x_0, then its image $\varphi(M)$ is a k-manifold near $\varphi(x_0)$;

(b) M is a k-manifold if and only if $\varphi(M)$ is a k-manifold.

This applies, in particular, to shifts, rotations, and all invertible affine transformations of $\mathbb{R}^n$.

(4a2) $$S(E \cup F) = S(E) + S(F)$$

(4a3) $$\operatorname{vol}(E) \inf_{x \in E} f(x) \le S(E) \le \operatorname{vol}(E) \sup_{x \in E} f(x)$$

(4b1) $$f \text{ is bounded;} \quad \text{that is,} \quad \sup_{x \in \mathbb{R}^n} |f(x)| < \infty ,$$

(4b2) $$f \text{ has bounded support;} \quad \text{that is,} \quad \sup_{x : f(x) \ne 0} |x| < \infty .$$

(4b3) $$L_N(f) = \sum_{k \in \mathbb{Z}^n} L_{N,k}(f) , \quad L_{N,k}(f) = 2^{-nN} \inf_{x \in 2^{-N}(Q+k)} f(x) ,$$

(4b4) $$U_N(f) = \sum_{k \in \mathbb{Z}^n} U_{N,k}(f) , \quad U_{N,k}(f) = 2^{-nN} \sup_{x \in 2^{-N}(Q+k)} f(x) ;$$

here $Q = [0,1]^n$. Clearly, $L_N(f) \le U_N(f)$ and $L_N(f) = -U_N(-f)$.

4b5 Lemma. For every N, $L_{N+1}(f) \ge L_N(f)$, $\quad U_{N+1}(f) \le U_N(f)$.

$$L(f) = \lim_{N \to \infty} L_N(f) , \quad U(f) = \lim_{N \to \infty} U_N(f) .$$

Clearly, $-\infty < L(f) \le U(f) < \infty$.

4c6 Proposition (linearity)**.** All integrable functions $\mathbb{R}^n \to \mathbb{R}$ are a vector space, and the integral is a linear functional on this space.

4c7 Remark. Monotonicity: if $f(\cdot) \le g(\cdot)$ then $\displaystyle {}_*\!\!\int f \le {}_*\!\!\int g , \quad {}^*\!\!\int f \le {}^*\!\!\int g,$

and for integrable f, g, $\displaystyle \int f \le \int g .$

Homogeneity: $\displaystyle {}_*\!\!\int cf = c\, {}_*\!\!\int f , \quad {}^*\!\!\int cf = c\, {}^*\!\!\int f \quad \text{for } c \ge 0;$

$$ {}_*\!\!\int cf = c\, {}^*\!\!\int f , \quad {}^*\!\!\int cf = c\, {}_*\!\!\int f \quad \text{for } c \le 0;$$

if f is integrable then cf is, and $\int cf = c \int f$ for all $c \in \mathbb{R}$.

(Sub-, super-) additivity: $\displaystyle {}^*\!\!\int (f+g) \le {}^*\!\!\int f + {}^*\!\!\int g; \quad {}_*\!\!\int (f+g) \ge {}_*\!\!\int f + {}_*\!\!\int g;$

if f, g are integrable then $f + g$ is, and $\int (f+g) = \int f + \int g$.

$$v(E) = \int_{\mathbb{R}^n} \mathbb{1}_E ; \quad \int_E f = \int_{\mathbb{R}^n} f \cdot \mathbb{1}_E .$$

(4d6) $$\int_E 1 = v(E) ; \quad \int_E c = cv(E) \text{ for } c \in \mathbb{R} ;$$

(4d7) $$v(E) \inf_{x \in E} f(x) \le \int_E f \le v(E) \sup_{x \in E} f(x) ;$$

(4d8) $$v(E) = 0 \implies \int_E f = 0 .$$

$$\frac{1}{v(E)} \int_E f \qquad \text{the mean value of } f \text{ on } E .$$

$$\rho\big([f],[g]\big) = \| [f] - [g] \| = {}^*\!\!\int_B |f - g| ; \quad \text{the integral metric.}$$

We may safely ignore values of integrands on sets of volume zero (as far as they are bounded). Likewise we may ignore sets of volume zero when dealing with volume.

The set of all (equivalence classes of) integrable functions is closed (in the integral metric).

4f1 Exercise. (a) Every continuous $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable;

(b) every continuous function on a box is integrable on this box.

4f3 Proposition. Step functions are dense among integrable functions (in the integral metric).

4f5 Remark. The set of all (equivalence classes of) integrable functions is the closure of the set of all (equivalence classes of) step functions (in the integral metric).

4f7 Corollary. The set of all (equivalence classes of) integrable functions is the closure of the set of all (equivalence classes of) continuous functions with bounded support (in the integral metric).

4f9 Corollary. The (pointwise) product of two integrable functions is integrable.

4f14 Proposition. If $E, F \subset \mathbb{R}^n$ are admissible sets, then the sets $E \cap F$, $E \cup F$ and $E \setminus F$ are admissible.

4f16 Proposition. (a) A function integrable on $\mathbb{R}^n$ is integrable on every admissible set;

(b) a function integrable on an admissible set is integrable on every admissible subset of the given set.

4g6 Proposition. For every bounded $f:\mathbb{R}^n\to\mathbb{R}$ with bounded support,

$$ {}_*\!\!\int_{\mathbb{R}^n} f = \sup\Big\{ \int_{\mathbb{R}^n} g \,\Big|\, \text{step } g \le f \Big\}, \qquad {}^*\!\!\int_{\mathbb{R}^n} f = \inf\Big\{ \int_{\mathbb{R}^n} h \,\Big|\, \text{step } h \ge f \Big\}. $$

4g7 Corollary. For every bounded $f:\mathbb{R}^n\to\mathbb{R}$ with bounded support,

$$ {}_*\!\!\int_{\mathbb{R}^n} f = \sup\Big\{ \int_{\mathbb{R}^n} g \,\Big|\, \text{integrable } g \le f \Big\}, \qquad {}^*\!\!\int_{\mathbb{R}^n} f = \inf\Big\{ \int_{\mathbb{R}^n} h \,\Big|\, \text{integrable } h \ge f \Big\}. $$

4g8 Corollary. A function $f:\mathbb{R}^n\to\mathbb{R}$ is integrable if and only if for every $\varepsilon>0$ there exist step functions g and h such that $g\le f\le h$ and $\int_{\mathbb{R}^n} h - \int_{\mathbb{R}^n} g \le \varepsilon$.

4g9 Exercise. (c) for every integrable function $f:\mathbb{R}^n\to\mathbb{R}$ and $\varepsilon>0$ there exist continuous functions g and h with bounded support such that $g\le f\le h$ and $\int_{\mathbb{R}^n} h - \int_{\mathbb{R}^n} g \le \varepsilon$.

4g10 Exercise. (b) additivity of the upper integral: ${}^*\!\int_{E\uplus F} f = {}^*\!\int_E f + {}^*\!\int_F f$, and the same for the lower integral.

(c) (4d7) holds for lower and upper integrals.

4h1 Proposition. $f(\cdot+a)$ is integrable if and only if f is integrable, and in this case $\int_{\mathbb{R}^n} f(\cdot+a) = \int_{\mathbb{R}^n} f$.

4h2 Corollary. For every set $E\subset\mathbb{R}^n$ and vector $a\in\mathbb{R}^n$, the shifted set $E+a$ is admissible if and only if E is admissible, and in this case $v(E+a)=v(E)$.

(4h5) $$ |a|^n \int_{\mathbb{R}^n} f(ax)\,\mathrm{d}x = \int_{\mathbb{R}^n} f\,, $$

(4h6) $$ v(aE) = |a|^n v(E)\,. $$

4h7 Proposition. For every integrable $f:\mathbb{R}^n\to\mathbb{R}$ and $\varepsilon>0$ there exists $\delta>0$ such that for all $a\in\mathbb{R}^n$

$$ |a|\le\delta \quad\Longrightarrow\quad \|f(\cdot+a)-f\|\le\varepsilon\,. $$

4i1 Proposition. If a function $f:\mathbb{R}^n\to[0,\infty)$ is integrable, then the set $E=\{(x,t):0<t<f(x)\}\subset\mathbb{R}^n\times\mathbb{R}$ is admissible, and $\quad v_{n+1}(E)=\int_{\mathbb{R}^n} f$.

4i2 Corollary. If functions $f,g:\mathbb{R}^n\to\mathbb{R}$ are integrable, then the set $E=\{(x,t):f(x)<t<g(x)\}\subset\mathbb{R}^n\times\mathbb{R}$ is admissible.

4i3 Exercise. For f as in 4i1, the set $\{(x,t):t=f(x)>0\}\subset\mathbb{R}^n\times\mathbb{R}$ is of volume zero.

(5b1) $$ \int_{\mathbb{R}^n}\Big(y\mapsto\int_{\mathbb{R}^m} f(\cdot,y)\Big) = \int_{\mathbb{R}^{m+n}} f = \int_{\mathbb{R}^m}\Big(x\mapsto\int_{\mathbb{R}^n} f(x,\cdot)\Big) $$

for every step function $f:\mathbb{R}^{m+n}\to\mathbb{R}$. The same holds for every continuous f with bounded support.

5b2 Exercise.

$$ \int_{\mathbb{R}^{m+n}} f(x_1,\dots,x_m)g(y_1,\dots,y_n)\,\mathrm{d}x_1\dots\mathrm{d}x_m\,\mathrm{d}y_1\dots\mathrm{d}y_n = $$
$$ = \Big(\int_{\mathbb{R}^m} f(x_1,\dots,x_m)\,\mathrm{d}x_1\dots\mathrm{d}x_m\Big)\Big(\int_{\mathbb{R}^n} g(y_1,\dots,y_n)\,\mathrm{d}y_1\dots\mathrm{d}y_n\Big) $$

for continuous functions $f:\mathbb{R}^m\to\mathbb{R}$, $g:\mathbb{R}^n\to\mathbb{R}$ with bounded support.

5c6 Exercise. Consider a function $f:\mathbb{R}^2\to\mathbb{R}$ of the form $f(x,y)=g(x)h(y)$ where $g,h:\mathbb{R}\to\mathbb{R}$ are bounded functions with bounded support.

(a) If g is negligible, then f is negligible.

(b) Integrability of f does not imply that the set $\{x: f(x,\cdot) \text{ is not integrable}\}$ is of volume zero.

5d1 Theorem. If a function $f:\mathbb{R}^{m+n}\to\mathbb{R}$ is integrable, then the iterated integrals

$$ \int_{\mathbb{R}^m}\mathrm{d}x\ {}_*\!\!\int_{\mathbb{R}^n}\mathrm{d}y\,f(x,y)\,, \qquad \int_{\mathbb{R}^m}\mathrm{d}x\ {}^*\!\!\int_{\mathbb{R}^n}\mathrm{d}y\,f(x,y)\,, $$
$$ \int_{\mathbb{R}^n}\mathrm{d}y\ {}_*\!\!\int_{\mathbb{R}^m}\mathrm{d}x\,f(x,y)\,, \qquad \int_{\mathbb{R}^n}\mathrm{d}y\ {}^*\!\!\int_{\mathbb{R}^m}\mathrm{d}x\,f(x,y) $$

are well-defined and equal to $\iint_{\mathbb{R}^{m+n}} f(x,y)\,\mathrm{d}x\mathrm{d}y$.

Clarification. The claim that $\int\mathrm{d}x\,{}_*\!\int\mathrm{d}y\,f(x,y)$ is well-defined means that the function $x\mapsto{}_*\!\int\mathrm{d}y\,f(x,y)$ is integrable.

The equality

$$ \int\Big(x\mapsto{}_*\!\!\int f(x,\cdot)\Big) = \int\Big(x\mapsto{}^*\!\!\int f(x,\cdot)\Big) $$

implies integrability (with the same integral) of every function sandwiched between the lower and upper integrals. It is convenient to interpret $x\mapsto\int f(x,\cdot)$ as *any* such function and write, as before,

$$ \int_{\mathbb{R}^{m+n}} f = \int_{\mathbb{R}^m}\Big(x\mapsto\int_{\mathbb{R}^n} f(x,\cdot)\Big) $$

and

$$ \int\mathrm{d}x\int\mathrm{d}y\,f(x,y) = \iint f(x,y)\,\mathrm{d}x\mathrm{d}y = \int\mathrm{d}y\int\mathrm{d}x\,f(x,y) $$

even though f_x may be non-integrable for some x.

5d3 Exercise. 5b2 generalizes to integrable functions

(a) assuming integrability of the function $(x,y)\mapsto f(x)g(y)$,

(b) deducing integrability of this function from integrability of f and g (via sandwich).

5e1 Exercise. If $E_1\subset\mathbb{R}^m$ and $E_2\subset\mathbb{R}^n$ are admissible sets then the set $E=E_1\times E_2\subset\mathbb{R}^{m+n}$ is admissible.

5e2 Corollary. Let $f:\mathbb{R}^{m+n}\to\mathbb{R}$ be integrable on every box, and $E\subset\mathbb{R}^{m+n}$ admissible set; then

$$ \int_E f = \int_{\mathbb{R}^m}\Big(x\mapsto\int_{E_x} f_x\Big) \quad \text{where } E_x=\{y:(x,y)\in E\}\subset\mathbb{R}^n \text{ for } x\in\mathbb{R}^m\,. $$

(5e3) $$ v_{m+n}(E) = \int_{\mathbb{R}^m} v_n(E_x)\,\mathrm{d}x \qquad \text{where } v_k \text{ is the volume in } \mathbb{R}^k; $$

for instance, the volume of a 3-dimensional geometric body is the 1-dimensional integral of the area of the 2-dimensional section of the body.

5e7 Exercise. For f, g and E as in 4i2

(a) $v_{n+1}(E)=\int_{\mathbb{R}^n}(g-f)^+$;

(b) $\int_E h = \int_{\mathbb{R}^n}\mathrm{d}x\,\mathbb{1}_{f<g}(x)\int_{f(x)}^{g(x)}\mathrm{d}t\,h(x,t) \quad$ for every $h:E\to\mathbb{R}$ integrable on E.

5e8 Remark. Here $\mathbb{1}_{f<g}$ is the indicator of the set $\{x:f(x)<g(x)\}$. This set need not be admissible. And nevertheless, the iterated integral is well-defined (according to the clarifications...).

5e14 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function; then

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n\, f(x_n) = \int_0^x f(t) \frac{(x-t)^{n-1}}{(n-1)!}\, dt\,.$$

5e18 Exercise. Suppose the function f depends only on the first coordinate. Then

$$\int_V f(x_1)\, dx = v_{n-1} \int_{-1}^1 f(x_1)(1-x_1^2)^{(n-1)/2}\, dx_1\,,$$

where V is the unit ball in $\mathbb{R}^n$, and v_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

(6b1) $$\mathrm{Osc}_f(x_0) = \inf_{r>0} \mathrm{Osc}_f\big(\{x : |x-x_0| < r\}\big)\,,$$

where $\mathrm{Osc}_f(U) = \mathrm{diam} f(U) = \sup_{x\in U} f(x) - \inf_{x\in U} f(x)$.

6b2 Theorem.

$$\int^*_{\mathbb{R}^n} f - \int_{*\,\mathbb{R}^n} f = \int^*_{\mathbb{R}^n} \mathrm{Osc}_f\,.$$

6b5 Lemma (Lebesgue's covering number)**.** Let $K \subset \mathbb{R}^n$ be a compact set, $U_1, \dots, U_m \subset \mathbb{R}^n$ open sets, and $K \subset U_1 \cup \dots \cup U_m$. Then

$$\exists \delta > 0\ \forall x \in K\ \exists i \in \{1,\dots,m\}\ \forall y\ \big(|y-x| < \delta \implies y \in U_i\big)\,.$$

6b7 Corollary. A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if Osc_f is negligible.

6b8 Exercise. For a set $E \subset \mathbb{R}^n$,

(a) $\mathrm{Osc}_{\mathbf{1}_E} = \mathbb{1}_{\partial E}$;
(b) E is admissible if and only if ∂E has volume 0;
(c) $v^*(E) - v_*(E) = v^*(\partial E)$;
(d) if E is admissible, then E° and $\overline{E}$ are admissible, and $v(E^\circ) = v(E) = v(\overline{E})$.

6b9 Exercise. For sets $E, F \subset \mathbb{R}^n$,

(a) $\partial(E \cup F) \subset \partial E \cup \partial F$, $\partial(E \cap F) \subset \partial E \cup \partial F$, $\partial(E \setminus F) \subset \partial E \cup \partial F$.

(6b12) $$(f \text{ is integrable on } E) \iff (\mathrm{Osc}_f \text{ is negligible on } E^\circ)\,.$$

6c2 Proposition. Countable union of sets of measure 0 has measure 0.

6c3 Proposition. A compact set has measure 0 if and only if it has volume 0.

6c4 Exercise. (a) If Z has measure 0, then $Z^\circ = \varnothing$, and $v_*(Z) = 0$.

6d2 Theorem (Lebesgue's criterion)**.** A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if it is continuous almost everywhere.

6d3 Lemma. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. If f is negligible then $f(\cdot) = 0$ almost everywhere.

6d4 Lemma. The set $\{x : \mathrm{Osc}_f(x) \ge \varepsilon\}$ is compact, for every $\varepsilon > 0$.

7a1 Proposition. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then, for every $E \subset \mathbb{R}^n$,

$$A(E) \text{ is admissible} \iff E \text{ is admissible.}$$

7a2 Lemma. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then, for every set $Z \subset \mathbb{R}^n$ of volume 0, the set $A(Z)$ has volume 0.

> The notion "admissible set" is insensitive to a change of basis.
> This notion is well-defined in every n-dimensional vector space, and preserved by isomorphisms of these spaces.

The same holds for the notion "volume 0".

7b1 Proposition. If a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ preserves the Euclidean metric, then it preserves volume.

> Volume is insensitive to a change of orthonormal basis. It is well-defined in every n-dimensional Euclidean space, and preserved by isomorphisms of these spaces.

7b3 Theorem. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then, for every admissible $E \subset \mathbb{R}^n$,

$$v\big(A(E)\big) = |\det A|\, v(E)\,.$$

> On an n-dimensional vector space the volume is ill defined, but admissibility is well defined, and the ratio $\frac{v(E_1)}{v(E_2)}$ of volumes is well defined. That is, the volume is well defined up to a coefficient.

7c1 Theorem. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then, for every bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,

$$|\det A| \int_* f \circ A = \int_* f \qquad \text{and} \qquad |\det A| \int^* f \circ A = \int^* f\,.$$

Thus, $f \circ A$ is integrable if and only if f is integrable, and in this case

$$|\det A| \int f \circ A = \int f\,.$$

8a1 Theorem. Let $U, V \subset \mathbb{R}^n$ be admissible open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$ a bounded function such that the function $(f \circ \varphi)|\det D\varphi| : U \to \mathbb{R}$ is also bounded. Then (a) $(f$ is integrable on $V) \iff (f \circ \varphi$ is integrable on $U) \iff ((f \circ \varphi)|\det D\varphi|$ is integrable on $U)$;

(b) if they are integrable, then $\int_V f = \int_U (f \circ \varphi)|\det D\varphi|$.

8b2 Exercise (polar coordinates in $\mathbb{R}^2$)**.** (a)

$$\int_{x^2+y^2<R^2} f(x,y)\, \mathrm{d}x\mathrm{d}y = \int_{0<r<R,\, 0<\theta<2\pi} f(r\cos\theta, r\sin\theta)\, r\, \mathrm{d}r\mathrm{d}\theta$$

for every integrable function f on the disk $x^2 + y^2 < R^2$.

8b3 Exercise (spherical coord. in $\mathbb{R}^3$)**.** $\Psi(r,\varphi,\theta) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$;

(c) $|\det D\Psi| = r^2 \sin\theta$.

8b9 Proposition (the second Pappus's centroid theorem)**.** Let $\Omega \subset (0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$ be an admissible set and $\tilde\Omega = \{(x,y,z) : \big(\sqrt{x^2+y^2}, z\big) \in \Omega\} \subset \mathbb{R}^3$. Then $\tilde\Omega$ is admissible, and $v_3(\tilde\Omega) = v_2(\Omega) \cdot 2\pi x_{C_\Omega}$; here $C_\Omega = (x_{C_\Omega}, z_{C_\Omega})$ is the centroid of Ω.

8c1 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets, and $\varphi : U \to V$ diffeomorphism. Then, for every set $Z \subset U$,

$$(Z \text{ has measure } 0) \iff (\varphi(Z) \text{ has measure } 0)\,.$$

8c4 Lemma. Let $E \subset \mathbb{R}^n$ be an admissible set, and $f : E \to \mathbb{R}$ a bounded function. Then f is integrable on E if and only if the discontinuity points of f on E° are a set of measure 0.

8c5 Corollary. A set $E \subset U$ is admissible if and only if $\varphi(E) \subset V$ is admissible.

(9b1) $$\int_G f = \sup \Big\{ \int_{\mathbb{R}^n} g \Big| g : \mathbb{R}^n \to \mathbb{R} \text{ integrable, } 0 \le g \le f \text{ on } G,\, g = 0 \text{ on } \mathbb{R}^n \setminus G \Big\}.$$

(9b4) $$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}\,. \quad \text{Poisson formula}$$

9b9 Proposition. $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2 \in [0, \infty]$ for all $f_1, f_2 \ge 0$ on G, continuous almost everywhere.

9b10 Proposition (exhaustion)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,
$$G_k \uparrow G \implies \int_{G_k} f \Big\uparrow \int_G f \in [0, \infty]$$
for all $f : G \to [0, \infty)$ continuous almost everywhere.

9b11 Corollary (monotone convergence for volume)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,
$$G_k \uparrow G \implies v_*(G_k) \uparrow v_*(G)\,.$$

(9c1) $$\Gamma(t) = \int_0^\infty x^{t-1} \mathrm{e}^{-x}\, \mathrm{d}x \quad \text{for } t \in (0, \infty)\,.$$

(9c2) $$\Gamma(t+1) = t\Gamma(t) \quad \text{for } t \in (0, \infty)\,.$$

(9c3) $$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \dots$$

(9c4) $$\int_0^\infty x^a \mathrm{e}^{-x^2}\, \mathrm{d}x = \frac{1}{2}\Gamma\Big(\frac{a+1}{2}\Big) \quad \text{for } a \in (-1, \infty)\,,$$

(9c5) $$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}\,.$$

(9c6) $$\Gamma\Big(\frac{2n+1}{2}\Big) = \frac{1}{2} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{2n-1}{2} \sqrt{\pi}\,.$$

(9c7) $$V_n = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})}\,. \quad \text{volume of the n-dimensional unit ball}$$

(9c8) $$\int_0^{\pi/2} \cos^{\alpha-1}\theta \sin^{\beta-1}\theta\, \mathrm{d}\theta = \frac{1}{2} \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})} \quad \text{for } \alpha, \beta \in (0, \infty)\,.$$

(9c9) $$\int_0^{\pi/2} \sin^{\alpha-1}\theta\, \mathrm{d}\theta = \int_0^{\pi/2} \cos^{\alpha-1}\theta\, \mathrm{d}\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\big(\frac{\alpha}{2}\big)}{\Gamma\big(\frac{\alpha+1}{2}\big)}\,.$$

(9c10) $$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, \mathrm{d}x = \mathrm{B}(\alpha, \beta) \quad \text{for } \alpha, \beta \in (0, \infty)\,,$$

(9c11) $$\mathrm{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha, \beta \in (0, \infty)$$

9d1 Theorem (change of variables)**.** Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to [0, \infty)$. Then

(a) (f is continuous almost everywhere on V) $\iff$ ($f \circ \varphi$ is continuous almost everywhere on U) $\iff$(($f \circ \varphi$)$|\det D\varphi|$ is continuous almost everywhere on U);

(b) if they are, then $\displaystyle\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \in [0, \infty]$.

(9e1) $$\int_{*G} f = \sup \Big\{ \int_{\mathbb{R}^n} g \Big| g : \mathbb{R}^n \to \mathbb{R} \text{ integrable, } 0 \le g \le f \text{ on } G,\, g = 0 \text{ on } \mathbb{R}^n \setminus G \Big\}.$$
In particular, if f is continuous almost everywhere on G, then ${}_*\int_G f = \int_G f$.

9e4 Proposition. If $g_1, g_2, \dots : \mathbb{R}^n \to [0, \infty)$ are integrable and $g_k \uparrow f : \mathbb{R}^n \to [0, \infty)$, then $\int_{\mathbb{R}^n} g_k \uparrow {}_*\int_{\mathbb{R}^n} f$.

9e5 Theorem (monotone convergence for Riemann integral)**.** If $g, g_1, g_2, \dots : \mathbb{R}^n \to \mathbb{R}$ are integrable and $g_k \uparrow g$, then $\int_{\mathbb{R}^n} g_k \uparrow \int_{\mathbb{R}^n} g$.

9e6 Theorem (iterated improper integral)**.** If a function $f : G \to [0, \infty)$ is continuous almost everywhere, then
$${}_*\int_{\mathbb{R}^m} \mathrm{d}x\, {}_*\int_{G_x} \mathrm{d}y\, f(x, y) = \iint_G f(x, y)\, \mathrm{d}x\mathrm{d}y \in [0, \infty]\,.$$

9e7 Corollary. The volume of an open set $G \subset \mathbb{R}^{m+n}$ is equal to the lower integral of the volume of G_x (even if G is not admissible).

9f1 Proposition.
$$\underset{\substack{x_1, \dots x_n > 0,\\ x_1 + \dots + x_n < 1}}{\int \!\!\cdots\!\! \int} x_1^{p_1-1} \dots x_n^{p_n-1}\, \mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1 + \dots + p_n + 1)} \quad \text{for all } p_1, \dots p_n > 0\,.$$

Volume of the unit ball in the metric l_p: $\quad v\big(B_p(1)\big) = \dfrac{2^n\Gamma^n\big(\frac{1}{p}\big)}{p^n\Gamma\big(\frac{n}{p}+1\big)}\,.$

For improperly integrable $f : G \to \mathbb{R}$ (that is, continuous a.e. and $\int_G |f| < \infty$):

(9g3) $$\int_G f = \int_G f^+ - \int_G f^-\,.$$

9g4 Exercise. Linearity: $\int_G cf = c\int_G f$ for $c \in \mathbb{R}$, and $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2$.

9g5 Proposition (exhaustion)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,
$$G_k \uparrow G \implies \int_{G_k} f \to \int_G f \in \mathbb{R}\,.$$

9g6 Theorem (change of variables)**.** Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$. Then

(a) (f is continuous almost everywhere on V) $\iff$ ($f \circ \varphi$ is continuous almost everywhere on U) $\iff$(($f \circ \varphi$)$|\det D\varphi|$ is continuous almost everywhere on U);

(b) if they are, then $\displaystyle\int_V |f| = \int_U |(f \circ \varphi) \det D\varphi| \in [0, \infty]$;

(c) and if the integrals in (b) are finite, then $\displaystyle\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \in \mathbb{R}$.

9g7 Exercise. (a) The function $t \mapsto \int_0^\infty x^{t-1}\mathrm{e}^{-x} \ln x\, \mathrm{d}x$ is continuous on $(0, \infty)$;

(b) the gamma function is continuously differentiable on $(0, \infty)$, and
$$\Gamma'(t) = \int_0^\infty x^{t-1}\mathrm{e}^{-x} \ln x\, \mathrm{d}x \quad \text{for } 0 < t < \infty\,;$$

(c) the gamma function is convex on $(0, \infty)$.