5 Pushforward, pullback, and change of variables

Change of variables need not be one-to-one, which is surprisingly useful for the integral of Jacobian, divergence, and even topology.

5a Pushforward and pullback: introduction

5a1 Definition. (a) Let $M \subset \mathbb{R}^N$ be a manifold (of some dimension n). A mapping $\varphi : M \to \mathbb{R}^{N_2}$, $\varphi(x) = \big(\varphi_1(x), \dots, \varphi_{N_2}(x)\big)$, is *continuously differentiable*, in symbols $\varphi \in C^1(M \to \mathbb{R}^{N_2})$, if $\varphi_1, \dots, \varphi_{N_2} \in C^1(M)$.[1]

(b) Let $M_1 \subset \mathbb{R}^{N_1}$, $M_2 \subset \mathbb{R}^{N_2}$ be manifolds (of some dimensions n_1, n_2). A mapping $\varphi : M_1 \to M_2$ is *continuously differentiable*, in symbols $\varphi \in C^1(M_1 \to M_2)$, if φ is continuously differentiable as a mapping $M_1 \to \mathbb{R}^{N_2}$. If, in addition, φ is invertible and $\varphi^{-1} \in C^1(M_2 \to M_1)$, then φ is a *diffeomorphism* $M_1 \to M_2$.[2]

5a2 Exercise. If (G, ψ) is a chart of an n-dimensional manifold $M \subset \mathbb{R}^N$, then ψ is a diffeomorphism between the n-dimensional manifold $G \subset \mathbb{R}^n$ and the n-dimensional manifold $\psi(G) \subset M \subset \mathbb{R}^N$.

Prove it.[3]

5a3 Exercise. Let $U, V \subset \mathbb{R}^N$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $M \subset U$ a manifold. Then the set $\varphi(M) \subset V$ is a manifold, and $\varphi|_M : M \to \varphi(M)$ is a diffeomorphism.

Prove it.

The set $C^1(M)$ is an algebra (recall 2b11); the set $C^1(M \to \mathbb{R}^{N_2})$ is a vector space; $C^1(M_1 \to M_2)$ is not.

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[1] Recall 2b10.

[2] If a diffeomorphism exists, M_1 and M_2 are called diffeomorphic. The condition $n_1 = n_2$ is necessary and not sufficient.

[3] Hint: recall 2a9, 2b11.

5a4 Exercise. If $\varphi \in C^1(M_1 \to M_2)$ and $\psi \in C^1(M_2 \to M_3)$, then $\psi \circ \varphi \in C^1(M_1 \to M_3)$.

Prove it.[1]

We'll see soon that some mathematical objects related to M_1 may be transferred to M_2 via a given $\varphi \in C^1(M_1 \to M_2)$; this is "pushforward", denoted by φ_*. Some objects may be transferred from M_2 to M_1; this is "pullback", denoted by φ^*. Sometimes φ is required to be of class C^2. And some objects may be transferred by diffeomorphisms only (in both directions, since φ^{-1} is also a diffeomorphism). Remarkably, the following universal relations hold in all cases:

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_* , \quad (\psi \circ \varphi)^* = \varphi^* \circ \psi^* . \tag{5a5}$$

Points: pushforward. A point $x \in M_1$ leads to the point $\varphi(x) \in M_2$. That is, $\varphi_*(x) = \varphi(x)$ for $x \in M_1$. But a point $y \in M_2$ does not lead to a point of M_1 (unless φ is invertible); the inverse image $\{x : \varphi(x) = y\}$ may contain more than one point, and may be empty.

Functions: pullback. A function $f \in C^1(M_2)$ leads to the function $f \circ \varphi \in C^1(M_1)$. That is, $\varphi^*(f) = f \circ \varphi$ for $f \in C^1(M_2)$. But a function $f \in C^1(M_1)$ does not lead to a function on M_2 (unless φ is invertible).

Note that φ^* is linear on $C^1(M_2)$. A preserved relation:

$$\underset{\text{for } x \in M_1, f \in C^1(M_2)\,.}{f\big(\varphi_*(x)\big) = \big(\varphi^*(f)\big)(x)} \qquad \begin{gathered} x_1 \overset{\varphi_*}{\longmapsto} x_2,\ f_1 \overset{\varphi^*}{\longleftarrow\!\shortmid} f_2 \\ f_1(x_1) = f_2(x_2) \end{gathered}$$

Paths: pushforward. A path $\gamma \in C^1([t_0, t_1] \to M_1)$ leads to the path $\varphi \circ \gamma \in C^1([t_0, t_1] \to M_2)$. That is, $\varphi_*(\gamma) = \varphi \circ \gamma$.

A preserved relation:

$$\underset{\text{for } t \in [t_0, t_1], \gamma \in C^1([t_0, t_1] \to M_1)\,.}{\big(\varphi_*(\gamma)\big)(t) = \varphi_*\big(\gamma(t)\big)} \qquad \begin{gathered} \gamma_1 \overset{\varphi_*}{\longmapsto} \gamma_2,\ x_1 \overset{\varphi_*}{\longmapsto} x_2 \\ \gamma_1(t) = x_1 \quad \Longrightarrow \quad \gamma_2(t) = x_2 \end{gathered}$$

Universal relations (5a5) hold evidently in the three cases treated above.

Tangent vectors: pushforward. It is easy to guess that a vector $h \in T_{x_1} M_1$ leads to the vector $\varphi_*(h) \in T_{x_2} M_2$ where $x_2 = \varphi(x_1)$, and a preserved relation holds:

$$\underset{\text{for } t \in [a, b], \gamma \in C^1([a, b] \to M_1)\,.}{\varphi_*\big(\gamma'(t)\big) = \big(\varphi_*(\gamma)\big)'(t)} \qquad \begin{gathered} \gamma_1 \overset{\varphi_*}{\longmapsto} \gamma_2,\ h_1 \overset{\varphi_*}{\longmapsto} h_2 \\ \gamma_1'(t) = h_1 \quad \Longrightarrow \quad \gamma_2'(t) = h_2 \end{gathered}$$

[1] Hint: M_2 is locally a graph.

But note that the chain rule (from Analysis-3) does not apply to $\varphi \circ \gamma$, since φ is defined on M_1, and M_1 is not open (unless $n_1 = N_1$), thus, $D\varphi$ is undefined. Note also that the notation $\varphi_*(h) \in T_{x_2}M_2$ is flawed; rather, it should be $\varphi_*(x_1, h_1) = (x_2, h_2)$, where $x_2 = \varphi(x_1)$ and $h_2 \in T_{x_2}M_2$.

5a6 Definition. The *tangent bundle* TM of an n-manifold $M \subset \mathbb{R}^N$ is the set

$$TM = \{(x, h) : x \in M,\, h \in T_xM\} \subset \mathbb{R}^{2N}\,.$$

5a7 Example. Let $M = \{\big(t, f(t)\big) : t \in \mathbb{R}\}$ be the graph of a function $f \in C^1(\mathbb{R})$; then (recall 2b20)

$$TM = \{\big(t, f(t), \lambda, \lambda f'(t)\big) : t, \lambda \in \mathbb{R}\} \subset \mathbb{R}^4\,.$$

If in addition $f \in C^2(\mathbb{R})$, then TM is a 2-manifold covered by a single chart $\mathbb{R}^2 \ni (t, \lambda) \mapsto \big(t, f(t), \lambda, \lambda f'(t)\big)$. Otherwise this mapping is a homeomorphism (think, why) but not a diffeomorphism.

5a8 Exercise. If (G, ψ) is a chart of M, then the mapping

$$(u, v) \mapsto \big(\psi(u), (D\psi)_u v\big)$$

is a homeomorphism from $G \times \mathbb{R}^n$ onto a relatively open subset of TM.

Prove it.[1]

5a9 Lemma. Let $M_1 \subset \mathbb{R}^{N_1}$, $M_2 \subset \mathbb{R}^{N_2}$ be manifolds (of some dimensions n_1, n_2), and $\varphi \in C^1(M_1 \to M_2)$. Then there exists one and only one mapping $D\varphi \in C(TM_1 \to TM_2)$ such that

$$\big((\varphi \circ \gamma)(t), (\varphi \circ \gamma)'(t)\big) = (D\varphi)\big(\gamma(t), \gamma'(t)\big)$$

whenever $\gamma \in C^1([t_0, t_1] \to M_1)$ is a path, and $t \in [t_0, t_1]$.

Proof. Given $x_1 \in M_1$, we consider a chart (G, ψ) of M_1 around x_1, and the corresponding C^1 mapping (not just chart) $\xi = \varphi \circ \psi : G \to M_2$. Let γ_1 be a path in M_1 such that $\gamma_1(0) = x_1$, and $\gamma_2 = \varphi \circ \gamma_1$ the corresponding path in M_2; clearly, $\gamma_2(0) = x_2 = \varphi(x_1)$. Assuming that γ_1 does not escape $\psi(G)$ (otherwise restrict γ_1 to a smaller interval of t) we introduce the path $\beta = \psi^{-1} \circ \gamma_1$ in G and note that $\gamma_1 = \psi \circ \beta$, $\gamma_2 = \xi \circ \beta$ (since $\gamma_2 = \varphi \circ \gamma_1 = \varphi \circ \psi \circ \beta = \xi \circ \beta$). It follows that $\gamma_1'(0) = (D\psi)_{\beta(0)}\beta'(0)$ and $\gamma_2'(0) = (D\xi)_{\beta(0)}\beta'(0)$, therefore

$$\gamma_2'(0) = (D\xi)_{\beta(0)}\ (D\psi)_{\beta(0)}\big)^{-1}\gamma_1'(0)\,.$$

[1] Hint: $|(D\psi)_{u_2}h_2 - (D\psi)_{u_1}h_1| \ge |(D\psi)_{u_2}(h_2 - h_1)| - \|(D\psi)_{u_2} - (D\psi)_{u_1}\| \cdot |h_1|$, and $|(D\psi)_{u_2}(h_2 - h_1)| \ge |h_1 - h_2| / \|\big((D\psi)_{u_2}\big)^{-1}\|$.

Uniqueness of $D\varphi$ follows: it must be

$$(D\varphi)(x_1,h_1) = (x_2,h_2) \quad \text{where } x_2 = \varphi(x_1) \text{ and } h_2 = (D\xi)_{\psi^{-1}(x_1)}\big((D\psi)_{\psi^{-1}(x_1)}\big)^{-1}h_1\,, \tag{5a10}$$

since for every $h_1 \in T_{x_1}M_1$ there exists a path γ_1 in M_1 such that $\gamma_1(0) = x_1$ and $\gamma_1'(0) = h_1$ (recall 2b19 and try a linear path β).

Locally, existence of $D\varphi$ is ensured by (5a10); continuity follows via 5a8 from continuity of the mapping $(u,v) \mapsto \big(\xi(u),(D\xi)_u v\big)$. For two charts, the corresponding local mappings agree on the intersection (by the uniqueness). Glued together, these local mappings give $D\varphi$. $\square$

It is tempting to say that $\varphi = \xi\circ\psi^{-1}$ and therefore $D\varphi = (D\xi)\circ(D(\psi^{-1}))$. Really, it is; but this fact does not follow from the chain rule (of Analysis-3).

It is convenient to write $h_2 = (D\varphi)_x h_1$ or $h_2 = (D_{h_1}\varphi)_x$ instead of $(\varphi(x),h_2) = (D\varphi)(x,h_1)$.

Note that the mapping $(D\varphi)_x : T_xM_1 \to T_{\varphi(x)}M_2$ is linear.

So, $\varphi_* = D\varphi$ on TM; $\varphi_*(x,h) = (D\varphi)(x,h) = \big(\varphi(x),(D\varphi)_x h\big)$. The relevant universal relation $(\psi\circ\varphi)_* = \psi_*\circ\varphi_*$ (recall (5a5)) holds for tangent bundles, which follows from the corresponding relation for paths (think, why). It means that

$$D(\psi\circ\varphi)_x h = (D\psi)_{\varphi(x)}(D\varphi)_x h\,, \tag{5a11}$$

the chain rule of Analysis-4!

Every $f \in C^1(M)$ may be treated as a C^1 mapping from M to the 1-dimensional manifold $\mathbb{R}$; in this case $(Df)_x : T_xM \to \mathbb{R}$, thus, Df is a 1-form on M.

Differential 1-forms: pullback. A 1-form ω on M_2 leads to the 1-form $\varphi^*(\omega)$ on M_1 defined by

$$\big(\varphi^*(\omega)\big)(x,h) = \omega\big(\varphi_*(x),\varphi_*(h)\big) = \omega\big(\varphi(x),(D\varphi)_x h\big)\,.$$

In order to get $\varphi^*(\omega) \in C^1$ one needs not only $\omega \in C^1$ but also $\varphi \in C^2$.

Note that φ^* is linear on the vector space of 1-forms on M_2.

Preserved relations:

$$\varphi^*(f\omega) = \varphi^*(f)\varphi^*(\omega) \quad \text{for } f\in C(M_2) \text{ and 1-form } \omega \text{ on } M_2\,. \qquad \begin{gathered} f_1 \xleftarrow{\varphi^*} f_2,\ \omega_1 \xleftarrow{\varphi^*} \omega_2 \\ f_1\omega_1 \xleftarrow{\varphi^*} f_2\omega_2 \end{gathered} \tag{5a12}$$

$$D(\varphi^* f) = \varphi^*(Df) \quad \text{for } f\in C^1(M_2)\,. \qquad \begin{gathered} f_1 \xleftarrow{\varphi^*} f_2,\ \omega_1 \xleftarrow{\varphi^*} \omega_2 \\ Df_2 = \omega_2 \implies Df_1 = \omega_1 \end{gathered} \tag{5a13}$$

Relation (5a12) follows immediately from the definition of $\varphi^*(\omega)$. Relation (5a13) follows from the chain rule (5a11): $\big(D(\varphi^* f)\big)(x,h) = D(f\circ\varphi)_x h = (Df)_{\varphi(x)}(D\varphi)_x h = \big(\varphi^*(Df)\big)(x,h)$.

Treating a path γ in M as a C^1 mapping from the 1-dimensional manifold $(t_0,t_1)\subset\mathbb{R}$ to M, we introduce the 1-form $\gamma^*(\omega)$ on (t_0,t_1) and observe that $\gamma^*(\omega)$ is equal to the volume form on (t_0,t_1) multiplied by the function $t\mapsto\omega\big(\gamma(t),\gamma'(t)\big)$, whence

$$\int_{(t_0,t_1)} \gamma^*(\omega) = \int_{t_0}^{t_1} \omega\big(\gamma(t),\gamma'(t)\big)\,\mathrm{d}t\,.$$

We see that a pullback lurks in the definition (1c10) of $\int_\gamma \omega$:

$$\int_\gamma \omega = \int_{(t_0,t_1)} \gamma^*(\omega)\,. \tag{5a14}$$

We get another preserved relation:

$$\int_\gamma \varphi^*(\omega) = \int_{\varphi_*(\gamma)} \omega \qquad\qquad \begin{gathered} \gamma_1 \overset{\varphi_*}{\longmapsto} \gamma_2,\ \omega_1 \overset{\varphi^*}{\longleftarrow\!\shortmid} \omega_2 \\ \textstyle\int_{\gamma_1}\omega_1 = \int_{\gamma_2}\omega_2 \end{gathered} \tag{5a15}$$

whenever γ is a path in M_1, ω is a 1-form on M_2, and $\varphi\in C^1(M_1\to M_2)$. Proof: by (5a14), $\int_\gamma \varphi^*(\omega) = \int_{(t_0,t_1)} \gamma^*(\varphi^*(\omega))$; and, using (5a5),

$$\int_{\varphi_*(\gamma)} \omega = \int_{\varphi\circ\gamma} \omega = \int_{(t_0,t_1)} (\varphi\circ\gamma)^*(\omega) = \int_{(t_0,t_1)} \gamma^*(\varphi^*(\omega))\,.$$

Singular boxes: pushforward. Similarly to paths, $\varphi_*(\Gamma)=\varphi\circ\Gamma$ for a singular box $\Gamma : B\to M_1$.

Differential n-forms: pullback. Similarly to 1-forms,

$$\begin{aligned}\big(\varphi^*(\omega)\big)(x,h_1,\dots,h_n) = \omega\big(\varphi_*(x),\varphi_*(h_1),\dots,\varphi_*(h_n)\big) = \\ = \omega\big(\varphi(x),(D\varphi)_x h_1,\dots,(D\varphi)_x h_n\big)\,.\end{aligned}$$

In order to get $\varphi^*(\omega)\in C^1$ one needs not only $\omega\in C^1$ but also $\varphi\in C^2$.

Note that φ^* is linear on the vector space of n-forms on M_2.

A preserved relation: similarly to (5a12),

$$\begin{gathered}\varphi^*(f\omega) = \varphi^*(f)\varphi^*(\omega) \\ \text{for } f\in C(M_2) \text{ and } n\text{-form } \omega \text{ on } M_2\,.\end{gathered} \qquad\qquad \begin{gathered} f_1 \overset{\varphi^*}{\longleftarrow\!\shortmid} f_2,\ \omega_1 \overset{\varphi^*}{\longleftarrow\!\shortmid} \omega_2 \\ f_1\omega_1 \overset{\varphi^*}{\longleftarrow\!\shortmid} f_2\omega_2 \end{gathered} \tag{5a16}$$

Similarly to (5a14), now (1e12) becomes (think, why)

(5a17) $$\int_\Gamma \omega = \int_{B^\circ} \Gamma^*(\omega)\,.$$

A preserved relation: similarly to (5a15),

(5a18) $$\int_\Gamma \varphi^*(\omega) = \int_{\varphi_*(\Gamma)} \omega \qquad \begin{gathered} \Gamma_1 \xmapsto{\varphi_*} \Gamma_2,\ \omega_1 \xleftarrow{\varphi^*} \omega_2 \\ \textstyle\int_{\Gamma_1} \omega_1 = \int_{\Gamma_2} \omega_2 \end{gathered}$$

whenever Γ is a singular n-box in M_1, ω is an n-form on M_2, and $\varphi \in C^1(M_1 \to M_2)$.

Also, if $\varphi : M_1 \to M_2$ is an orientation preserving diffeomorphism between oriented n-dimensional manifolds $(M_1, \mathcal{O}_1)$, $(M_2, \mathcal{O}_2)$, and ω is an n-form on M_2, then we have another preserved relation: $\varphi^*\omega$ is integrable if and only if ω is integrable, and in this case

(5a19) $$\int_{(M_1,\mathcal{O}_1)} \varphi^*\omega = \int_{(M_2,\mathcal{O}_2)} \omega\,. \qquad \begin{gathered} M_1 \leftrightarrow M_2,\ \omega_1 \leftrightarrow \omega_2 \\ \textstyle\int_{M_1} \omega_1 = \int_{M_2} \omega_2 \end{gathered}$$

5a20 Exercise. Prove (5a19)
(a) for a single-chart ω;
(b) for a compactly supported ω;
(c) in general.[1]

What do you think about the relation $\int_{M_1} \varphi^* f = \int_{M_2} f$ for compactly supported $f \in C(M_2)$?

Vector fields: this is another story; see Sect. 5b.

When $\varphi : M_1 \to M_2$ is a diffeomorphism, it is convenient to define both pushforward and pullback in all cases; namely, when pushforward is already defined, we define pullback by $\varphi^* = (\varphi^{-1})_*$; and when pullback is already defined, we define pushforward by $\varphi_* = (\varphi^{-1})^*$. Two more relations $(\varphi_*)^{-1} = \varphi^*$, $(\varphi^*)^{-1} = \varphi_*$ follow from (5a5) and the universal relations $(\mathrm{id})_* = \mathrm{id}$, $(\mathrm{id})^* = \mathrm{id}$ that hold evidently in all cases. Here is how they follow: $\varphi^{-1} \circ \varphi = \mathrm{id} = \varphi \circ \varphi^{-1}$, therefore $(\varphi^{-1})_* \circ \varphi_* = \mathrm{id} = \varphi_* \circ (\varphi^{-1})_*$, that is, $(\varphi_*)^{-1} = (\varphi^{-1})_*$. Similarly, $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

For example: a path γ_1 in M_1 leads to the path $\gamma_2 = \varphi_*(\gamma_1) = \varphi \circ \gamma_1$ in M_2; and a path γ_2 in M_2 leads to the path $\gamma_1 = \varphi^*(\gamma_2) = (\varphi^{-1})_*(\gamma_2) = \varphi^{-1} \circ \gamma_2$ in M_1; and $\varphi^*\big(\varphi_*(\gamma_1)\big) = \varphi^{-1} \circ (\varphi \circ \gamma_1) = (\varphi^{-1} \circ \varphi) \circ \gamma_1 = \gamma_1$.

We'll often write $\varphi^* f$, $\varphi^* h$, $\varphi^* \omega$ etc. instead of $\varphi^*(f)$, $\varphi^*(h)$, $\varphi^*(\omega)$ etc.

[1] Hints: (a) similar to (5a17), use (2c2); (b) recall (2d4); (c) recall the paragraph before 2d7.

5b Vector fields: three facets of one notion

By a vector field on a manifold $M \subset \mathbb{R}^N$ one means (by default) a *tangent vector field,* that is, a mapping $F : M \to \mathbb{R}^N$ such that

$$\forall x \in M \;\; F(x) \in T_x M \,.$$

.... *Facet 1: velocity field* ....

Given two n-manifolds $M_1 \subset \mathbb{R}^{N_1}$, $M_2 \subset \mathbb{R}^{N_2}$, a diffeomorphism $\varphi : M_1 \to M_2$, and a vector field F of class C^0 on M_1, one may define the vector field $\varphi_* F$ of class C^0 on M_2 by

$$\text{(5b1)} \quad (\varphi_* F)(y) = \varphi_*\big(F(\varphi^*(y))\big) = \varphi_*\big(F(\varphi^{-1}(y))\big) = \\ = (D\varphi)_{\varphi^{-1}(y)}\big(F(\varphi^{-1}(y))\big) = (D\varphi)_x\big(F(x)\big) \quad \text{where } x = \varphi^{-1}(y)$$

for $y \in M_2$.

5b2 Exercise. If a path γ_1 in M_1 conforms to a vector field F_1 on M_1 in the sense that

$$\forall t \in [t_0, t_1] \;\; \gamma_1'(t) = F_1\big(\gamma(t)\big) \,,$$

then the path $\gamma_2 = \varphi_*(\gamma_1)$ in M_2 conforms (in the same sense) to the vector field $F_2 = \varphi_*(F_1)$ on M_2.

Prove it.

We see that the transfer (5b1) is appropriate when vector fields are interpreted as velocity fields.

5b3 Exercise (polar coordinates)**.** Let $M_1 = (0, \infty) \times (-\pi, \pi)$, $M_2 = \mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}$ (treated as 2-dimensional manifolds in $\mathbb{R}^2$), $\varphi : M_1 \to M_2$, $\varphi\big(\begin{smallmatrix} r \\ \theta \end{smallmatrix}\big) = \big(\begin{smallmatrix} r\cos\theta \\ r\sin\theta \end{smallmatrix}\big)$. Then the relation $F_2 = \varphi_* F_1$ (or equivalently $F_1 = \varphi^* F_2$) between vector fields F_1 on M_1 and F_2 on M_2 becomes

$$F_1 \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} F_2 \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} .$$

Prove it.

In particular, a radial vector field $F_2\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big) = g\big(\sqrt{x^2+y^2}\big)\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big)$ corresponds to $F_1\big(\begin{smallmatrix} r \\ \theta \end{smallmatrix}\big) = \big(\begin{smallmatrix} rg(r) \\ 0 \end{smallmatrix}\big)$. Taking $g(r) = 1/r^2$ we have $\operatorname{div} F_2 = 0$ (recall (4a7)), and $F_1\big(\begin{smallmatrix} r \\ \theta \end{smallmatrix}\big) = \big(\begin{smallmatrix} 1/r \\ 0 \end{smallmatrix}\big)$, $\operatorname{div} F_1 \ne 0$.

If puzzled, recall the footnote on page 66: divergence 0 means preservation of volume, not mass. The diffeomorphism φ does not preserve the area (the 2-dimensional volume).

In contrast to numerous good news in Sect. 5a, now we face bad news: the relation $\operatorname{div} F = f$ is not equivalent to $\operatorname{div}(\varphi^* F) = \varphi^* f$. Also, the flux of F through a boundary is not preserved by diffeomorphisms.

Vector fields are nice to visualize, but not nice to transform.

. . . . *Facet 2: gradient; visualization of* 1*-forms* . . . .

Recall the gradient $\nabla f \in C^0(U \to \mathbb{R}^n)$ of a function $f \in C^1(U)$ on an open set $U \subset \mathbb{R}^n$; ∇f is a vector field, generally not interpreted as a velocity field. It is related to the 1-form $Df : (x,h) \mapsto (Df)_x h$ by $Df(x,h) = \langle \nabla f(x), h \rangle$. More generally, every 1-form ω on U corresponds to a vector field F on U such that $\omega(x,h) = \langle F(x), h \rangle$.

What about the gradient of a function $f \in C^1(M)$ on an n-dimensional manifold $M \subset \mathbb{R}^N$? We may define it by $(f \circ \gamma)'(t) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle$ for all paths γ in M and all $t \in [t_0, t_1]$ and prove existence and uniqueness. But we already have the 1-form Df on M. We may define ∇f by $Df(x,h) = \langle \nabla f(x), h \rangle$ for all $x \in M$, $h \in T_x M$. For each x the vector $\nabla f(x) \in T_x M$ is thus well-defined, but is it continuous in x? And can we express it via a chart? Yes; in fact,

$$\nabla f\big(\psi(u)\big) = (D\psi)_u \big((D\psi)_u^{\mathrm{t}} (D\psi)_u \big)^{-1} \nabla (f \circ \psi)(u) \, ; \tag{5b4}$$

here $(D\psi)_u^{\mathrm{t}} (D\psi)_u$ is the matrix $\big(\langle (D_i\psi)_u, (D_j\psi)_u \rangle \big)_{i,j}$ seen before (in Sect. 2c; the root from its determinant was denoted by $J_\psi(u)$). The same approach may be used for representing a given 1-form ω by a vector field F such that $\omega(x,h) = \langle F(x), h \rangle$.

5b5 Exercise (polar coordinates)**.** Let M_1, M_2 and φ be as in 5b3.

(a) Prove, without using (5b4), that $\nabla(f \circ \varphi) = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \big((\nabla f) \circ \varphi \big)$ for all $f \in C^1(M_2)$;

(b) check that $(D\psi)^{\mathrm{t}} (D\psi) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$. Does (5b4) hold in this case (for $\psi = \varphi$)?

Denoting $F_1 = \nabla(f \circ \varphi)$, $F_2 = \nabla f$ we have $F_1 \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} F_2 \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$, which is not the same as 5b3 (but for radial vector fields they are the same).

We see that the transfer (5b1) is inappropriate when vector fields are interpreted as gradients (or, more generally, visualize 1-forms).

. . . . *Facet 3: visualization of* $(n-1)$*-forms; flux* . . . .

Recall the linear one-to-one correspondence (4e6) between $(N-1)$-forms on $\mathbb{R}^N$ and (continuous) vector fields on $\mathbb{R}^N$. More generally, we may introduce a linear one-to-one correspondence between $(n-1)$-forms ω on an

n-dimensional oriented manifold $(M, \mathcal{O})$ in $\mathbb{R}^N$ and (tangent, continuous) vector fields F on M by

$$\omega(x, h_1, \dots, h_{n-1}) = \mu(x, F(x), h_1, \dots, h_{n-1})$$

whenever $h_1, \dots, h_{n-1} \in T_x M$; here μ is the volume form on $(M, \mathcal{O})$. But for now we remain in the framework of (4e6); $n = N - 1$.

Recall also the adjugate matrix (Sect. 0f).

$n = N - 1$

5b6 Lemma. Let $U_1, U_2 \subset \mathbb{R}^N$ be open sets; $\varphi \in C^1(U_1 \to U_2)$; and $F_1 \in C(U_1 \to \mathbb{R}^N)$, $F_2 \in C(U_2 \to \mathbb{R}^N)$ the vector fields that correspond to n-forms ω_1, ω_2 such that $\omega_1 = \varphi^* \omega_2$. Then $F_1 = (\operatorname{adj} D\varphi)(F_2 \circ \varphi)$, that is,

$$F_1(x) = \operatorname{adj}(D\varphi)_x F_2(\varphi(x)) \quad \text{for all } x \in U_1 .$$

$$\begin{tikzcd} \omega_1 \arrow[d, leftrightarrow] & \omega_2 \arrow[l, mapsto, "\varphi^*"'] \arrow[d, leftrightarrow] \\ F_1 & F_2 \arrow[l, mapsto, "\varphi^*"'] \end{tikzcd}$$

Proof. Denote for convenience $A = (D\varphi)_x$, $B = \operatorname{adj} A$, $v_1 = F_1(x)$, $v_2 = F_2(\varphi(x))$; we have to prove that $v_1 = Bv_2$.

The relation $\omega_1 = \varphi^* \omega_2$ at x, in terms of v_1, v_2, becomes

$$\forall h_1, \dots, h_n \in \mathbb{R}^N \quad \det(v_1, h_1, \dots, h_n) = \det(v_2, Ah_1, \dots, Ah_n) .$$

It is sufficient to prove that $\det(v_1, h_1, \dots, h_n) = \det(Bv_2, h_1, \dots, h_n)$, that is,

$$\det(v_2, Ah_1, \dots, Ah_n) = \det(Bv_2, h_1, \dots, h_n) ,$$

just an algebraic equality.

For fixed v_1, v_2 and $h_1, \dots, h_n$ we treat both sides as functions of a matrix A. These functions being continuous (and moreover, polynomial, of degree $\le n$), we may restrict ourselves to invertible matrices A. Introducing $h_0 = A^{-1} v_2$ we have

$$\det(v_2, Ah_1, \dots, Ah_n) = \det(Ah_0, Ah_1, \dots, Ah_n) = \\ = (\det A) \det(h_0, h_1, \dots, h_n) ,$$

since the product of A by the matrix with the columns $h_0, \dots, h_n$ is the matrix with the columns $Ah_0, \dots, Ah_n$ (think, why). Finally, $(\det A) h_0 = (\det A) A^{-1} v_2 = Bv_2$. $\square$

Still another transfer (different from 5b3 and 5b5 even for radial vector fields, see the next exercise).

5b7 Exercise (polar coordinates)**.** Let M_1, M_2 and φ be as in 5b3, 5b5. Check that

$$F_1 \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} F_2 \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} .$$

5b8 Exercise (rotation)**.** Let $\varphi = L : \mathbb{R}^N \to \mathbb{R}^N$ be a linear transformation such that $\forall x \in \mathbb{R}^N \;\; |Lx| = |x|$, and $\det L = +1$. Then the relation $F_1 = (\operatorname{adj} D\varphi)(F_2 \circ \varphi)$ becomes

$$F_1 = L^{-1} \circ F_2 \circ L \, .$$

Prove it.

5b9 Proposition. For the constant vector field $F_2(x) = (1, 0, \dots, 0)$ and arbitrary mapping $\varphi : x \mapsto \big(\varphi_1(x), \dots, \varphi_N(x)\big)$ of class C^1, the corresponding vector field $F_1 = (\operatorname{adj} D\varphi)(F_2 \circ \varphi)$ is $\nabla\varphi_2 \times \dots \times \nabla\varphi_N$; that is,

$$F_1(x) = \nabla\varphi_2(x) \times \dots \times \nabla\varphi_N(x) \, .$$

5b10 Lemma. For every $N \times N$ matrix A, the first column of the matrix $\operatorname{adj} A$ is $a_2 \times \dots \times a_N$ where $a_1, \dots, a_N \in \mathbb{R}^N$ are the rows of A.[1]

Proof. Denote the first column of the matrix $\operatorname{adj} A$ by b_1. By the Laplace expansion (recall Sect. 0f), $\langle a_1, b_1 \rangle = \det A$. On the other hand, $\langle a_1, a_2 \times \dots \times a_N \rangle = \det A$. Also, b_1 does not depend on a_1 (mind the minors). Thus, $\langle a_1, b_1 \rangle = \langle a_1, a_2 \times \dots \times a_N \rangle$ for all a_1, which implies $b_1 = a_2 \times \dots \times a_N$. □

***Proof** of Prop. 5b9.* The rows of the matrix $A = (D\varphi)_x$ are $\nabla\varphi_1(x), \dots, \nabla\varphi_N(x)$. The vector $(\operatorname{adj} A)(1, 0, \dots, 0)$ is the first column of $\operatorname{adj} A$; by Lemma 5b10 it is $\nabla\varphi_2 \times \dots \times \nabla\varphi_N$. □

5b11 Corollary (of 5b9)**.** For the vector field $F_2(x_1, \dots, x_N) = (x_1, 0, \dots, 0)$ and arbitrary mapping $\varphi : x \mapsto \big(\varphi_1(x), \dots, \varphi_N(x)\big)$ of class C^1, the corresponding vector field $F_1 = (\operatorname{adj} D\varphi)(F_2 \circ \varphi)$ is $\varphi_1 \nabla\varphi_2 \times \dots \times \nabla\varphi_N$; that is,

$$F_1(x) = \varphi_1(x) \nabla\varphi_2(x) \times \dots \times \nabla\varphi_N(x) \, .$$

5b12 Corollary (of 5b6, 5a19 and 5a3)**.** Let $U_1, U_2 \subset \mathbb{R}^N$ be open sets, $\varphi : U_1 \to U_2$ a diffeomorphism, $\det D\varphi > 0$, F_2 a continuous vector field on U_2, and V_2 a smooth set such that $\overline{V}_2 \subset U_2$. Then $V_1 = \varphi^{-1}(V_2)$ is a smooth set such that $\overline{V}_1 \subset U_1$, $F_1 : x \mapsto \operatorname{adj}(D\varphi)_x F_2(\varphi(x))$ is a continuous vector field on U_1, and

$$\int_{\partial V_1} \langle F_1, \mathbf{n}_1 \rangle = \int_{\partial V_2} \langle F_2, \mathbf{n}_2 \rangle \, .$$

[1]Similarly, the i-th column of $\operatorname{adj} A$ is $(-1)^{i-1} a_1 \times \dots \times a_{i-1} \times a_{i+1} \times \dots \times a_N$.

5b13 Remark. But if $\det D\varphi < 0$, then $\int_{\partial V_1} \langle F_1, \mathbf{n}_1 \rangle = -\int_{\partial V_2} \langle F_2, \mathbf{n}_2 \rangle$; think, why. In general, U_1 decomposes in two disjoint open sets...

5b14 Remark. Keeping in mind possible applications to the piecewise smooth case, consider a bounded regular open (not necessarily smooth) set V_2 such that $\overline{V}_2 \subset U_2$, and a closed set $Z_2 \subset \partial V_2$ such that $\partial V_2 \setminus Z_2$ is an n-manifold of finite n-dimensional volume. Then $V_1 = \varphi^{-1}(V_2)$ is a bounded regular open set such that $\overline{V}_1 \subset U_1$, $Z_1 = \varphi^{-1}(Z_2) \subset \partial V_1$ is a closed set such that $\partial V_1 \setminus Z_1 = \varphi^{-1}(\partial V_2 \setminus Z_2)$ is an n-manifold of finite n-dimensional volume, and

$$\int_{\partial V_1 \setminus Z_1} \langle F_1, \mathbf{n}_1 \rangle = \int_{\partial V_2 \setminus Z_2} \langle F_2, \mathbf{n}_2 \rangle$$

for every continuous vector field F_2 on U_2; here $F_1 = (\operatorname{adj} D\varphi)(F_2 \circ \varphi)$. This is similar to 5b12.

5b15 Example. Find the flux of the radial vector field $F(x) = x$, $x \in \mathbb{R}^2$, through the cardioid $(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$.

We turn to polar coordinates.

The curve: $(r^2 - 2r\cos\theta)^2 = 4r^2$; $r^2 - 2r\cos\theta = \pm 2r$; $r = 2(\pm 1 + \cos\theta)$; $r = 2(1 + \cos\theta)$ for $-\pi < \theta < \pi$.

The vector field: $F_1\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} r^2 \\ 0 \end{pmatrix}$.

The flux, via (4e6): $\int_{-\pi}^{\pi} \det\Big(F_1\begin{pmatrix} r(\theta) \\ \theta \end{pmatrix}, \begin{pmatrix} r'(\theta) \\ 1 \end{pmatrix}\Big)\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \begin{vmatrix} r^2(\theta) & r'(\theta) \\ 0 & 1 \end{vmatrix}\,\mathrm{d}\theta = \int_{-\pi}^{\pi} 4(1 + \cos\theta)^2\,\mathrm{d}\theta = 12\pi$.

Here is an important preserved relation, in two versions.

5b16 Corollary (of 5b12 and Sect. 0c)**.** Let $U_1, U_2 \subset \mathbb{R}^N$ be open sets, $\varphi : U_1 \to U_2$ a diffeomorphism, F_2 a continuous vector field on U_2, and $f_2 \in C(U_2)$. If

$$\int_{V_2} f_2 = \int_{\partial V_2} \langle F_2, \mathbf{n}_2 \rangle \quad \text{for all smooth sets } V_2 \text{ such that } \overline{V}_2 \subset U_2\,,$$

then

$$\int_{V_1} f_1 = \int_{\partial V_1} \langle F_1, \mathbf{n}_1 \rangle \quad \text{for all smooth sets } V_1 \text{ such that } \overline{V}_1 \subset U_1\,,$$

where $F_1 = (\operatorname{adj} D\varphi)(F_2 \circ \varphi)$ and $f_1 = (\det D\varphi)(f_2 \circ \varphi)$.

5b17 Remark. No need to require $\det D\varphi > 0$, since the negative $\det D\varphi$ leads to $-\int_{V_1} f_1 = -\int_{\partial V_1} \langle F_1, \mathbf{n}_1 \rangle$.

5b18 Corollary (of 5b16 and 4a3)**.** Let $U_1, U_2, \varphi, F_1, F_2, f_1, f_2$ be as in 5b16, and in addition, $\varphi \in C^2$, $F_2 \in C^1$; then also $F_1 \in C^1$, and

$$f_1 = \operatorname{div} F_1 \quad \text{if and only if} \quad f_2 = \operatorname{div} F_2 \,.$$

$$\operatorname{div} F_1 = f_1 \quad \overset{F_1 \leftrightarrow F_2,\, f_1 \leftrightarrow f_2}{\iff} \quad \operatorname{div} F_2 = f_2$$

5b19 Exercise. Let U_1, U_2, φ be as in 5b18, $\varphi : x \mapsto \big(\varphi_1(x), \dots, \varphi_N(x)\big)$. Then

(a) $\operatorname{div}\big(\nabla\varphi_2 \times \dots \times \nabla\varphi_N\big) = 0$;

(b) $\operatorname{div}\big(\varphi_1 \nabla\varphi_2 \times \dots \times \nabla\varphi_N\big) = \det(D\varphi)$ (the Jacobian of φ).

Prove it.[1]

5b20 Exercise. Let $U_1, U_2 \subset \mathbb{R}^N$ be open sets, and $\varphi : U_1 \to U_2$ a diffeomorphism of class C^2. Let V_1 be a bounded regular open set, $\overline{V}_1 \subset U_1$, and $Z_1 \subset \partial V_1$ a closed set such that the divergence theorem holds for V_1 and $\partial V_1 \setminus Z_1$ (as defined by 4b4). Then the same holds for $V_2 = \varphi(V_1)$ and $Z_2 = \varphi(Z_1)$.

Prove it.

5b21 Exercise. (a) Consider the truncated cone (conical frustum) $V = \{(x,y,z) : a < z < b,\ x^2 + y^2 < cz^2\} \subset \mathbb{R}^3$ for given $a, b, c > 0$, $a < b$. Prove that the divergence theorem holds for V and $\partial V \setminus Z$ where $Z = \{(x,y,a) : x^2 + y^2 = ca^2\} \cup \{(x,y,b) : x^2 + y^2 = cb^2\}$.

(b) Consider the cone $V = \{(x,y,z) : 0 < z < b,\ x^2 + y^2 < cz^2\} \subset \mathbb{R}^3$ for given $b, c > 0$. Prove that the divergence theorem holds for V and $\partial V \setminus Z$ where $Z = \{(x,y,b) : x^2 + y^2 = cb^2\} \cup \{(0,0,0)\}$.[2]

5c Not just one-to-one[3]

Interestingly, 5b16 and 5b18 can be generalized to mappings φ that are not one-to-one. This generalization leads to divergence theorem for singular cubes, and ultimately, to Stokes' theorem. Surprisingly, main ideas may be demonstrated without vector fields (and differential forms), proving (3a3) and in addition, some famous topological results!

[1] Hint: 5b18, 5b9, 5b11.

[2] (b) take $a \to 0$ in (a).

[3] Based on: J. Milnor (1978) 'Analytic proofs of the "Hairy ball theorem" and the Brouwer fixed point theorem', Amer. Math. Monthly **85** 521–524;
C.A. Rogers (1980) "A less strange version of Milnor's proof…", Amer. Math. Monthly **87** 525–527;
K. Gröger (1981) "A simple proof of the Brouwer fixed point theorem", Math. Nachr. **102** 293–295.

The first idea is, to connect a given mapping with a diffeomorphism. We restrict ourselves to the simplest diffeomorphism id : $x \to x$ on a box or a smooth set.

5c1 Assumption. (a) $U \subset \mathbb{R}^n$ is either an open box or a smooth set;

(b) $\varphi \in C^1(\overline{U} \to \mathbb{R}^n)$, that is, $D\varphi$ extends to $\overline{U}$ by continuity (and therefore φ also extends to $\overline{U}$ by continuity).

5c2 Exercise. Prove that φ satisfies the Lipschitz condition:[1] there exists $L \in [0, \infty)$ such that

$$|\varphi(x) - \varphi(y)| \le L|x - y| \quad \text{for all } x, y \in \overline{U}\,.$$

We introduce

$$\text{(5c3)} \quad \varphi_t(x) = x + t\big(\varphi(x) - x\big) = (1-t)x + t\varphi(x) \quad \text{for } x \in \overline{U} \text{ and } t \in [0,1]\,.$$

Clearly, $\varphi_t \in C^1(\overline{U} \to \mathbb{R}^n)$ for each $t \in [0,1]$. It appears that φ_t must be a diffeomorphism for t small enough.

5c4 Lemma. There exists $\varepsilon \in (0,1]$ such that for every $t \in [0,\varepsilon]$ the mapping φ_t is a homeomorphism $\overline{U} \to \varphi_t(\overline{U})$, and $\varphi_t|_U$ is an orientation-preserving diffeomorphism $U \to \varphi_t(U)$.

Proof. First, using 5c2, $|\varphi_t(x) - \varphi_t(y)| = |(1-t)(x-y) + t(\varphi(x) - \varphi(y))| \ge (1-t)|x-y| - t|\varphi(x) - \varphi(y)| \ge (1-t)|x-y| - tL|x-y| = \big(1-(L+1)t\big)|x-y|$; for $t < 1/(L+1)$, φ_t is one-to-one and φ_t^{-1} is continuous on $\varphi_t(\overline{U})$, that is, φ_t is a homeomorphism.

Second, $\sup_{x \in \overline{U}} \|(D\varphi)_x\| = C < \infty$; $D\varphi_t = (1-t)I + tD\varphi$; $\|D\varphi_t - I\| = \| -tI + tD\varphi\| \le (C+1)t$; for $t < 1/(C+1)$, $\det D\varphi_t > 0$. By the inverse function theorem, φ_t is a local diffeomorphism. Being also a homeomorphism, it is a diffeomorphism. $\square$

5c5 Exercise. Let $U \subset \mathbb{R}^n$ be a bounded open set, $\psi : \overline{U} \to \mathbb{R}^n$ continuous. If $\psi(U)$ is open, then $\partial\big(\psi(U)\big) \subset \psi(\partial U)$.

Prove it.

5c6 Assumption. (c) U and $\mathbb{R}^n \setminus \overline{U}$ are connected (for a box this holds, of course);

(d) $\varphi(x) = x$ for all $x \in \partial U$.

[1]Hint: for a box U use convexity; for smooth U assume the contrary, choose $x_n \to x$, $y_n \to y$ such that $|\varphi(x) - \varphi(y)|/|x-y| \to \infty$ and note that $x = y$; in the case $x \in \partial U$ do similarly to the proof of 3b6.

5c7 Lemma. $\varphi_t(U) = U$ for all t small enough.

Proof. Denote $V = \mathbb{R}^n \setminus \overline{U}$ and $U_t = \varphi_t(U)$. For t small enough, by 5c4, U_t is open; by 5c5, $\partial U_t \subset \varphi_t(\partial U)$; also, $\varphi_t(\partial U) = \partial U$, and we get $\partial U_t \subset \partial U$.

We see that $\partial U_t \cap U = \emptyset$ and $\partial U_t \cap V = \emptyset$. By connectedness, $U_t \cap U$ is either $\emptyset$ or U, and $U_t \cap V$ is either $\emptyset$ or V. But U_t is bounded, while V is not. Thus, $U_t \cap V = \emptyset$, that is, $U_t \subset \overline{U}$; by regularity, $U_t \subset U$; and finally, $U_t = U$. □

The second idea is that

$$\text{the function} \quad t \mapsto \int_U \det D\varphi_t \quad \text{is a polynomial,} \tag{5c8}$$

since for every $x \in U$ the function $t \mapsto \det(D\varphi_t)_x = \det\big((1-t)I + t(D\varphi)_x\big)$ is a polynomial (of degree $\le n$). And if a polynomial is constant on some interval, then it is constant everywhere! Assuming 5c1 and 5c6 we have $\int_U \det D\varphi_t = v\big(\varphi_t(U)\big) = v(U)$ for all t small enough, therefore

$$\int_U \det D\varphi_t = v(U) \quad \text{for all } t \in \mathbb{R} \tag{5c9}$$

(but generally not equal to $v\big(\varphi_t(U)\big)$).

Now we are in position to prove (3a3).

5c10 Proposition.

$$\int_{\mathbb{R}^n} \det Df = 0 \quad \text{if } f \in C^1(\mathbb{R}^n \to \mathbb{R}^n) \text{ has a bounded support.}$$

Proof. We take $\varphi(x) = x + f(x)$, that is, $\varphi_t(x) = x + tf(x)$. We also take a "nice" U (say, a ball or a cube) such that f is compactly supported within U. Assumptions 5c1, 5c6 are satisfied. By (5c9), $\int_U \det D\varphi_t = v(U)$ for all t; $\int_U \det(I + tDf) = v(U)$; $\int_U \det\big(\frac{1}{t}I + Df\big) = \frac{1}{t^n}v(U)$; take the limit as $t \to \infty$. □

. . . . *A digression to topology* . . . .

We can also prove some famous topological results. First, a retraction theorem.

5c11 Proposition. For the unit ball $U = \{x : |x| < 1\} \subset \mathbb{R}^n$ there does not exist a mapping φ of class C^1 from $\overline{U}$ to ∂U such that $\forall x \in \partial U \;\; \varphi(x) = x$.

Proof. Such φ satisfies 5c1 and 5c6. By (5c9), $\int_U \det D\varphi_t = v(U)$ for all t. In particular, for $t = 1$ we get $\int_U \det D\varphi = v(U)$, which cannot happen, since $\varphi(U) \subset \partial U$ has empty interior, and therefore $\det D\varphi = 0$ everywhere. $\square$

Second, Brouwer fixed point theorem.

5c12 Proposition. For the unit ball $U = \{x : |x| < 1\} \subset \mathbb{R}^n$, every mapping $\varphi : \overline{U} \to \overline{U}$ of class C^1 has a fixed point (that is, $\exists x \in \overline{U} \;\; \varphi(x) = x$).

Proof. Otherwise, we define $\psi : \overline{U} \to \partial U$ by $\psi(x) = \varphi(x) + \lambda_x(x - \varphi(x))$ where $\lambda_x \ge 1$ is such that $|\psi(x)| = 1$, and apply 5c11 to ψ. $\square$

5c13 Remark. In topology, these facts are proved for continuous (rather than C^1) mappings. This is not our goal here, but anyway, a continuous $\varphi : \overline{U} \to \overline{U}$ may be approximated by $\varphi_k : \overline{U} \to \overline{U}$ of class C^1, then $\varphi_k(x_k) = x_k$, $x_k \to x$ (a subsequence...), and finally $\varphi(x) = x$.

Now, generalized 5c12 implies generalized 5c11: if φ is a retraction, then $(-\varphi)$ has no fixed point.

. . . .

Back to vector fields, pullbacks and differential forms.

In the rest of Sect. 5 we define the pullback of vector fields according to “facet 3” of 5b, that is,

$$\varphi^* F = (\operatorname{adj} D\varphi)(F \circ \varphi)\,. \tag{5c14}$$

We also redefine the pullback of functions (“scalar fields”) as

$$\varphi^* f = (\det D\varphi)(f \circ \varphi)\,. \tag{5c15}$$

That is, we treat F as a visualization of an $(N-1)$-form, and f as a visualization of an N-form $f \cdot \det$. Now 5b18 becomes preserved relation

$$f = \operatorname{div} F \quad \Longleftrightarrow \quad \varphi^* f = \operatorname{div}(\varphi^* F)\,,$$

$$\text{that is,} \qquad \varphi^*(\operatorname{div} F) = \operatorname{div}(\varphi^* F)$$

$$\begin{array}{ccc} f_1 & \longleftrightarrow & f_2 \\ {\scriptstyle\text{div}}\uparrow & & \uparrow{\scriptstyle\text{div}} \\ F_1 & \longleftrightarrow & F_2 \end{array}$$

provided that φ is a diffeomorphism of class C^2.

The notion of a polynomial $\mathbb{R}^N \to \mathbb{R}$ generalizes readily to the notion of a polynomial $\mathbb{R}^N \to \mathbb{R}^M$ or even $\mathbb{R}^N \to V$ where V is a finite-dimensional vector space. In particular, we may speak about polynomial vector fields $\mathbb{R}^N \to \mathbb{R}^N$.[1]

[1] For instance, $F\binom{x}{y} = \binom{x^{10}-5x^7+11}{-x^8}$.

On the other hand, we may speak about a polynomial family $(\varphi_t)_{t\in\mathbb{R}}$ of mappings $\varphi_t : \overline{U} \to \mathbb{R}^N$; in particular, (5c3) is such a family (of degree 1). Of course, $\varphi_t(x)$ is required to be polynomial in t, not in x.

5c16 Exercise. Let $(\varphi_t)_{t\in\mathbb{R}}$ be a polynomial family of mappings $\varphi_t \in C^1(\overline{U} \to \mathbb{R}^N)$. Then:

(a) For every polynomial $f : \mathbb{R}^N \to \mathbb{R}$, the family $(\varphi_t^* f)_t$ of functions on $\overline{U}$ is polynomial.

(b) For every polynomial vector fields $F : \mathbb{R}^N \to \mathbb{R}^N$, the family $(\varphi_t^* F)_t$ of vector fields on $\overline{U}$ is polynomial.
Prove it.

5c17 Proposition. Let $U \subset \mathbb{R}^N$ be an open set; $\varphi \in C^2(U \to \mathbb{R}^N)$; $F_2 : \mathbb{R}^N \to \mathbb{R}^N$ a polynomial vector field; $f_2 = \operatorname{div} F_2$; $f_1 = \varphi^* f_2$, and $F_1 = \varphi^* F_2$. Then $F_1 \in C^1(U \to \mathbb{R}^N)$, and

$$\operatorname{div} F_1 = f_1\,. \qquad \begin{tikzcd} f_1 & f_2 \arrow[l, mapsto, "\varphi^*"'] \\ F_1 \arrow[u, mapsto, "\operatorname{div}"] & F_2 \arrow[l, mapsto, "\varphi^*"] \arrow[u, mapsto, "\operatorname{div}"'] \end{tikzcd}$$

Proof. We introduce a polynomial family of mappings $\varphi_t \in C^2(U \to \mathbb{R}^N)$ by $\varphi_t(x) = x + (2-t)\big(\varphi(x) - x\big)$ and note that $\varphi_1(x) = \varphi(x)$, $\varphi_2(x) = x$. By 5c16, functions $f_t = \varphi_t^* f_2$ are a polynomial family. The notation is consistent: f_1, f_2 are as before. The same holds for vector fields $F_t = \varphi_t^* F_2$.

Clearly, $F_t \in C^1$ for all t; we'll prove that $\operatorname{div} F_t = f_t$ for all t. By the divergence theorem, $\int_V \operatorname{div} F_t = \int_{\partial V} \langle F_t, \mathbf{n}\rangle$ for every open ball[1] V such that $\overline{V} \subset U$. It is sufficient to prove that $\int_V f_t = \int_{\partial V} \langle F_t, \mathbf{n}\rangle$ for all such V (since two continuous functions with equal integrals over all balls must be equal). Let such ball V be given.

The function $t \mapsto \int_V f_t - \int_{\partial V} \langle F_t, \mathbf{n}\rangle$ being a polynomial, we may restrict ourselves to t close to 2. By 5c4, φ_t is an orientation-preserving diffeomorphism on a neighborhood of V. The set $V_t = \varphi_t(V)$ is smooth (since V is). By 5b12, $\int_{\partial V} \langle F_t, \mathbf{n}\rangle = \int_{\partial V_t} \langle F_2, \mathbf{n}\rangle$. By the change of variable theorem, $\int_V f_t = \int_{V_t} f_2$. The needed equality becomes $\int_{V_t} f_2 = \int_{\partial V_t} \langle F_2, \mathbf{n}\rangle$; the latter holds by the divergence theorem. □

5c18 Exercise. The formulas of 5b19, $\operatorname{div}\big(\nabla\varphi_2 \times \dots \times \nabla\varphi_N\big) = 0$ and $\operatorname{div}\big(\varphi_1 \nabla\varphi_2 \times \dots \times \nabla\varphi_N\big) = \det(D\varphi)$, hold for arbitrary $\varphi_1, \dots, \varphi_N$ of class C^2 (that is, φ need not be a diffeomorphism).

Prove it.[2]

---

[1] And moreover, for every smooth set V, of course.

[2] Hint: F_2 of 5b9, 5b11 are polynomial; use 5c17.

A wonder: on one hand, φ is required to be of class C^2, since otherwise F_1 need not be of class C^1 and $\operatorname{div} F_1$ need not exist; and on the other hand, second derivatives of φ do not occur in the formula $f_1 = (\det D\varphi)(f_2 \circ \varphi)$ for $\operatorname{div} F_1$!

We generalize Definition 3d3.

5c19 Definition. Let $U \subset \mathbb{R}^N$ be an open set, $F \in C(U \to \mathbb{R}^N)$ a vector field, and $f \in C(U)$ a function.[1] We say that f is the *generalized divergence* of F and write $f = \operatorname{div} F$, if

$$\int_V f = \int_{\partial V} \langle F, \mathbf{n} \rangle$$

for all smooth sets V such that $\overline{V} \subset U$.

5c20 Remark. (a) The generalized divergence is unique (that is, $f_1 = \operatorname{div} F$ and $f_2 = \operatorname{div} F$ imply $f_1 = f_2$);

(b) Def. 5c19 extends Def. 3d3; that is, if $F \in C^1$ then $\operatorname{tr}(DF)$ is the generalized divergence of F.

5c21 Example. In one dimension, a smooth set is a finite union of (separated) intervals (think, why); the relation $\int_V f = \int_{\partial V} \langle F, \mathbf{n} \rangle$ becomes just $\int_a^b f = F(b) - F(a)$; this equality (for all a, b such that $a < b$ and $[a, b] \subset U$) is necessary and sufficient for f to be the generalized divergence of F. If $F \in C^1$ then f exists and is the derivative, $f = F'$; and if $F \notin C^1$ then f does not exist.

We generalize Proposition 5c17.

5c22 Proposition. Let $U \subset \mathbb{R}^N$ be an open set; $\varphi \in C^1(U \to \mathbb{R}^N)$; $F_2 : \mathbb{R}^N \to \mathbb{R}^N$ a polynomial vector field; $f_2 = \operatorname{div} F_2$; $f_1 = \varphi^* f_2$, and $F_1 = \varphi^* F_2$. Then

$$f_1 = \operatorname{div} F_1 \,. \qquad \begin{array}{ccc} f_1 & \xleftarrow{\varphi^*} & f_2 \\ \uparrow{\scriptstyle \operatorname{div}} & & \uparrow{\scriptstyle \operatorname{div}} \\ F_1 & \xleftarrow{\varphi^*} & F_2 \end{array}$$

Note that "$f_2 = \operatorname{div} F_2$" may be interpreted classically, as $f_2 = \operatorname{tr}(DF_2)$, but "$f_1 = \operatorname{div} F_1$" is interpreted according to 5c19, since F_1 need not be of class C^1 (for a counterexample see 5c26 below).

5c23 Exercise. Prove Prop. 5c22.[2]

[1] Still more generally, one may consider an equivalence class of (locally) improperly integrable functions f.

[2] Hint: take the proof of Prop. 5c17 and throw away all unnecessary.

5c24 Exercise. Generalize 5c18 to $\varphi_1, \dots, \varphi_N$ of class C^1.

Prop. 5c22 is not yet a generalization of (the "if" part of) 5b18, since F_2 is required to be polynomial; but the next result is such generalization.

5c25 Proposition. Let $U, V \subset \mathbb{R}^N$ be open sets, $\varphi : U \to V$ a mapping[1]of class C^1, $F : V \to \mathbb{R}^N$ a vector field of class C^1. Then the generalized divergence of $\varphi^* F$ exists and is equal to $\varphi^*(\operatorname{div} F)$.

$$\begin{tikzcd} f_1 & f_2 \arrow[l, mapsto, "\varphi^*"'] \\ F_1 \arrow[u, mapsto, "\mathrm{div}"] & F_2 \arrow[l, mapsto, "\varphi^*"'] \arrow[u, mapsto, "\mathrm{div}"'] \end{tikzcd}$$

Proof. First, assume in addition that $F = \psi_1 \nabla\psi_2 \times \cdot \times \nabla\psi_N$ for some $\psi_1, \dots, \psi_N \in C^1(V)$. In this case we introduce the mapping $\psi : V \to \mathbb{R}^N$, $\psi(x) = \big(\psi_1(x), \dots, \psi_N(x)\big)$. By 5b11, $F = \psi^* G$ where $G : (x_1, \dots, x_N) \mapsto (x_1, 0, \dots, 0)$ is polynomial. Prop. 5c22 applies both to ψ and $\psi \circ \varphi$, giving $\psi^*(\operatorname{div} G) = \operatorname{div}(\psi^* G)$ and $(\psi \circ \varphi)^*(\operatorname{div} G) = \operatorname{div}\big((\psi \circ \varphi)^* G\big)$. Taking into account that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ we get $\psi^*(\operatorname{div} G) = \operatorname{div} F$ and $\varphi^*(\operatorname{div} F) = \operatorname{div}(\varphi^* F)$.

$$\begin{tikzcd} f_1 & f_2 \arrow[l, mapsto, "\varphi^*"'] & g \arrow[l, mapsto, "\psi^*"'] \\ F_1 \arrow[u, mapsto, "\mathrm{div}"] & F_2 \arrow[l, mapsto, "\varphi^*"'] \arrow[u, mapsto, "\mathrm{div}"'] & G \arrow[l, mapsto, "\psi^*"'] \arrow[u, mapsto, "\mathrm{div}"'] \end{tikzcd}$$

Second, if this claim holds for two vector fields, then it holds for their sum. It remains to prove that arbitrary F is the sum of some vector fields of the form $\psi_1 \nabla\psi_2 \times \cdot \times \nabla\psi_N$. We note that $F : x \mapsto \big(F_1(x), \dots, F_N(x)\big)$ is the sum of N "parallel" vector fields, the first being $x \mapsto \big(F_1(x), 0, \dots, 0\big)$, the last $x \mapsto \big(0, \dots, 0, F_N(x)\big)$. The first "parallel" vector field is $\psi_1 \nabla\psi_2 \times \cdot \times \nabla\psi_N$ where $\psi_1 = F_1$ and $\psi_k(x_1, \dots, x_N) = x_k$ for $k = 2, \dots, N$. Other "parallel" fields are treated similarly. $\square$

5c26 Example. Consider $\varphi \in C^1(\mathbb{R}^2 \to \mathbb{R}^2)$ of the form $\varphi\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big) = \big(\begin{smallmatrix} g(x) \\ y \end{smallmatrix}\big)$ for $g \in C^1(\mathbb{R})$, and the constant vector field $F_2(\cdot) = \big(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\big)$. We have $\operatorname{adj} D\varphi = \big(\begin{smallmatrix} 1 & 0 \\ 0 & g' \end{smallmatrix}\big)$;

$$F_1 \begin{pmatrix} x \\ y \end{pmatrix} = (\varphi^* F_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g'(x) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ g'(x) \end{pmatrix} .$$

By Prop. 5c25, such F_1 has the generalized divergence equal 0 (since $\operatorname{div} F_2 = 0$). Every $f \in C(\mathbb{R})$ is g' for some $g \in C^1(\mathbb{R})$, therefore

$$\operatorname{div} F = 0 \quad \text{for} \quad F : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ f(x) \end{pmatrix} , \quad f \in C(\mathbb{R}) .$$

We may rotate the plane (recall 5b8), getting

$$\operatorname{div} F = 0 \quad \text{for} \quad F : \begin{pmatrix} x \\ y \end{pmatrix} \to f(x\cos\theta + y\sin\theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} , \quad f \in C(\mathbb{R}) .$$

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[1]Note that $\varphi(U)$ need not be open.

The same holds for arbitrary linear combination of such vector fields (with different θ and f). Clearly, D_1F_1 and D_2F_2 are generally ill-defined, and nevertheless, $D_1F_1 + D_2F_2 = 0$ in some reasonable sense.

5d From smooth to singular

Recall the diffeomorphism invariance 5b20 of the notion "divergence theorem holds for V and $\partial V \setminus Z$" defined by 4b4; there, the equality $\int_V \operatorname{div} F = \int_{\partial V \setminus Z} \langle F, \mathbf{n} \rangle$ is required only for F continuously differentiable on $\overline{V}$. Now, what about the generalized divergence?

5d1 Proposition. Let $U, V \subset \mathbb{R}^N$ be open sets, $\overline{V} \subset U$, and $Z \subset \partial V$. If the divergence theorem holds for V and $\partial V \setminus Z$, and a vector field $F \in C(U \to \mathbb{R}^N)$ has the generalized divergence, then

$$\int_V \operatorname{div} F = \int_{\partial V \setminus Z} \langle F, \mathbf{n} \rangle \, .$$

The proof needs some preparations.

Given $f \in C(\mathbb{R}^N)$ and a box $B \subset \mathbb{R}^N$, we introduce $f_B : \mathbb{R}^N \to \mathbb{R}$ by

$$f_B(x) = \frac{1}{v(B)} \int_{B+x} f \, ;$$

that is, $f_B(x)$ is the mean value of f on the shifted box $B+x = \{b+x : b \in B\}$.

5d2 Exercise. (a) Let $N = 1$ and $B = [s,t]$. Prove that $f_B \in C^1(\mathbb{R})$ and $f'_B(x) = \frac{1}{t-s}\big(f(x+t) - f(x+s)\big)$.

(b) Let $N = 2$ and $B = [s_1,t_1] \times [s_2,t_2]$. Prove that $f_B \in C^1(\mathbb{R}^2)$ and

$$\frac{\partial}{\partial x_1} f_B(x_1,x_2) = \frac{1}{t_2 - s_2} \int_{[s_2,t_2]} \frac{1}{t_1 - s_1} \big(f(x_1{+}t_1, x_2{+}y) - f(x_1{+}s_1, x_2{+}y) \big) \, \mathrm{d}y \, .$$

(c) Prove that $f_B \in C^1(\mathbb{R}^N)$ in general.

5d3 Exercise. (a) For every $f \in C(\mathbb{R}^N)$ there exist $f_1, f_2, \dots \in C^1(\mathbb{R}^N)$ such that $f_k \to f$ (as $k \to \infty$) uniformly on bounded sets.

(b) Let $U \subset \mathbb{R}^N$ be an open set, and $f \in C(U)$. Then there exist open sets $U_k \uparrow U$ and functions $f_k \in C^1(U_k)$ such that $f_k \to f$ uniformly on compact subsets of U.

(c) The same holds for vector fields.

Prove it.[1]

[1] Hint: (a) consider f_B for a small B close to 0.

5d4 Exercise. Let a vector field $F \in C(\mathbb{R}^N \to \mathbb{R}^N)$ have the generalized divergence $\operatorname{div} F = f \in C(\mathbb{R}^N)$.

(a) For arbitrary $a \in \mathbb{R}^N$, the shifted vector field $F_a : x \mapsto F(x+a)$ and function $f_a : x \mapsto f(x+a)$ satisfy $\operatorname{div} F_a = f_a$.

(b) For arbitrary $a \in \mathbb{R}^N$ and $k = 1, 2, \dots$ the vector field $\tilde F = \frac{1}{k} \sum_{i=1}^k F_{\frac{i}{k}a}$ and function $\tilde f = \frac{1}{k} \sum_{i=1}^k f_{\frac{i}{k}a}$ satisfy $\operatorname{div} \tilde F = \tilde f$.

(c) For arbitrary $a \in \mathbb{R}^N$ the vector field $\tilde F = \int_0^1 F_{ta} \, \mathrm{d}t$ and function $\tilde f = \int_0^1 f_{ta} \, \mathrm{d}t$ satisfy $\operatorname{div} \tilde F = \tilde f$.

(d) For arbitrary box $B \subset \mathbb{R}^N$ the vector field $F_B : x \mapsto \frac{1}{v(B)} \int_{B+x} F$ and the function $f_B : x \mapsto \frac{1}{v(B)} \int_{B+x} f$ satisfy $\operatorname{tr}(DF_B) = \operatorname{div} F_B = f_B$.
Prove it.

5d5 Corollary (of 5d2–5d4)**.** Let $U \subset \mathbb{R}^N$ be an open set, and $F \in C(U \to \mathbb{R}^N)$ a vector field that has the generalized divergence. Then there exist open sets $U_k \uparrow U$ and vector fields $F_k \in C^1(U_k \to \mathbb{R}^N)$ such that $F_k \to F$ and $\operatorname{div} F_k \to \operatorname{div} F$ uniformly on compact subsets of U.

***Proof** of Prop. 5d1.* Corollary 5d5 gives us F_k. By the divergence theorem for V and $\partial V \setminus Z$ we have[1] $\int_V \operatorname{div} F_k = \int_{\partial V \setminus Z} \langle F_k, \mathbf{n} \rangle$, since $F_k \in C^1$. On the other hand, $\int_{\partial V \setminus Z} \langle F_k, \mathbf{n} \rangle \to \int_{\partial V \setminus Z} \langle F, \mathbf{n} \rangle$, since $F_k \to F$ uniformly on $\overline{V}$, and $v_n(\partial V \setminus Z) < \infty$ by 4d4. Also, $\int_V \operatorname{div} F_k \to \int_V \operatorname{div} F$, since $\operatorname{div} F_k \to \operatorname{div} F$ uniformly on $\overline{V}$. Thus, $\int_V f = \int_{\partial V \setminus Z} \langle F, \mathbf{n} \rangle$.[2] $\square$

We generalize Prop. 5c25.

5d6 Theorem. Let $U, V \subset \mathbb{R}^N$ be open sets, $\varphi : U \to V$ a mapping of class C^1, $F : V \to \mathbb{R}^N$ a vector field that has the generalized divergence. Then the generalized divergence of $\varphi^* F$ exists and is equal to $\varphi^*(\operatorname{div} F)$.

$$\begin{array}{ccc} f_1 & \xleftarrow{\ \varphi^*\ } & f_2 \\ {\scriptstyle \mathrm{div}}\uparrow & & \uparrow{\scriptstyle \mathrm{div}} \\ F_1 & \xleftarrow{\ \varphi^*\ } & F_2 \end{array}$$

5d7 Exercise. Prove Theorem 5d6.[3]

Let $B \subset \mathbb{R}^N$ be an open box; we know that the divergence theorem holds for B and $\partial B \setminus Z$; here $\partial B \setminus Z$ is the union of the $2N$ hyperfaces of B (and Z is the union of boxes of dimensions smaller than $N-1$), see 4b3 and the text after it.

[1] For k large enough.

[2] The fact that f_k are of class C^1 was not used; accordingly, we do not really need continuity of $\operatorname{div} F$; see the footnote to 5c19.

[3] Hint: similar to the proof of Prop. 5d1; pullback preserves the convergence uniform on compacta.

5d8 Theorem. Let a vector field $F \in C(U \to \mathbb{R}^N)$ on an open set $U \subset \mathbb{R}^N$ have the generalized divergence, and $\Gamma \in C^1(\overline{B} \to \mathbb{R}^N)$, $\Gamma(\overline{B}) \subset U$. Then

$$\int_B \Gamma^*(\operatorname{div} F) = \int_{\partial B \setminus Z} \langle \Gamma^*(F), \mathbf{n} \rangle \, .$$

Here Γ^* is interpreted according to (5c14), (5c15).

If Γ extends to a diffeomorphism on a neighborhood of $\overline{B}$, then $\int_{\partial B \setminus Z} \langle \Gamma^*(F), \mathbf{n} \rangle = \int_{\Gamma(\partial B \setminus Z)} \langle F, \mathbf{n} \rangle$ by 5b14, and $\int_B \Gamma^*(\operatorname{div} F) = \int_{\Gamma(B)} \operatorname{div} F$ by the change of variable theorem.

In general, Γ need not be one-to-one. Treating Γ as a singular box, one says that $\int_{\partial B \setminus Z} \langle \Gamma^*(F), \mathbf{n} \rangle$ is the flux of F through $\partial\Gamma$, and $\int_B \Gamma^*(\operatorname{div} F)$ is the integral of $\operatorname{div} F$ over Γ. Now 5d8 becomes the divergence theorem for a singular box.

Proof *of Theorem 5d8.* By Theorem 5d6, $\operatorname{div}(\Gamma^* F) = \Gamma^*(\operatorname{div} F)$ on B (generalized divergence). We exhaust B by smaller boxes: $B_1 \subset B_2 \subset \dots$, $\overline{B}_k \subset B$, $\cup_k B_k = B$. By Prop. 5d1, $\int_{\partial B_k \setminus Z_k} \langle \Gamma^* F, \mathbf{n} \rangle = \int_{B_k} \operatorname{div}(\Gamma^* F) = \int_{B_k} \Gamma^*(\operatorname{div} F)$; the limit as $k \to \infty$ completes the proof. □

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