Corrections to Sect. 2

To Sect. 2b

Exercise 2b7 should be formulated as follows:

The random continuous function $t \mapsto B(L+t) - B(L)$ is not a Brownian motion.

The hint to 2b7 fits to the formulation above. The claim of 2b7, as is, is also true, but needs more effort to prove.

To Sect. 2d

The proof of Lemma 2c3, given in Sect. 2d, is incomplete. The following additional argument is needed,

Given two Brownian motions B_1, B_2 on two probability spaces Ω_1, Ω_2 respectively, we construct a random continuous function Z on $\Omega = \Omega_1 \times \Omega_2$ as follows:

$$Z(t)(\omega_1, \omega_2) = \begin{cases} B_1(t)(\omega_1) & \text{if } t \le \tau_0(\omega_1), \\ B_1(\tau_0(\omega_1))(\omega_1) + B_2(t - \tau_0(\omega_1))(\omega_2) & \text{if } t \ge \tau_0(\omega_1); \end{cases}$$

here

$$\tau_0(\omega_1) = \begin{cases} 1 & \text{if } \max_{[0,1]} B_1(\cdot)(\omega_1) \ge 1, \\ \infty & \text{if } \max_{[0,1]} B_1(\cdot)(\omega_1) < 1. \end{cases}$$

Lemma A. The distribution of Z does not depend on the choice of B_1, B_2 (and Ω_1, Ω_2).

(Compare it with 2a4(b).)

Proof. Let B'_1, B'_2 be Brownian motions on Ω'_1, Ω'_2, and Z' constructed from B'_1, B'_2 in the same way as Z from B_1, B_2. We have to prove that

$$\big(Z'(t_1), \dots, Z'(t_j)\big) \sim \big(Z(t_1), \dots, Z(t_j)\big)$$

(identically distributed random vectors) for all j and $t_1, \dots, t_j \in [0, \infty)$. We may assume that $t_1 < \dots < t_j$ and $t_i = 1$ (for some i). We have

$$\big(Z(t_1), \dots, Z(t_j)\big) = f\big(\tau_0; B_1(t_1), \dots, B_1(t_j); B_2(t_{i+1} - 1), \dots, B_2(t_j - 1)\big)$$

for some $f : \{1, \infty\} \times \mathbb{R}^j \to \mathbb{R}^j$, namely,

$$f(1; x_1, \dots, x_j; y_{i+1}, \dots, y_j) = \big(x_1, \dots, x_i; x_i + y_{i+1}, \dots, x_i + y_j\big),$$
$$f(\infty; x_1, \dots, x_j; y_{i+1}, \dots, y_j) = \big(x_1, \dots, x_j\big).$$

Also,

$$\big(Z'(t_1),\dots,Z'(t_j)\big) = f\big(\tau'_0; B'_1(t_1),\dots,B'_1(t_j); B'_2(t_{i+1}-1),\dots,B'_2(t_j-1)\big)$$

(with the same f). Of course, f is Borel measurable. Thus, it is sufficient to prove that

$$\big(\tau_0; B_1(t_1),\dots,B_1(t_j); B_2(t_{i+1}-1),\dots,B_2(t_j-1)\big) \sim \\ \big(\tau'_0; B'_1(t_1),\dots,B'_1(t_j); B'_2(t_{i+1}-1),\dots,B'_2(t_j-1)\big)\,.$$

By independence, it is sufficient to prove that

$$\big(\tau_0; B_1(t_1),\dots,B_1(t_j)\big) \sim \big(\tau'_0; B'_1(t_1),\dots,B'_1(t_j)\big)\,,$$
$$\big(B_2(t_{i+1}-1),\dots,B_2(t_j-1)\big) \sim \big(B'_2(t_{i+1}-1),\dots,B'_2(t_j-1)\big)\,.$$

The latter is evident. The former is similar to 2a4(b). □

To Sect. 2f

Here we are in position to generalize Lemma A.

Lemma B. Let a function $T : C[0,\infty) \to [0,\infty]$ be measurable. ($C[0,\infty)$ is endowed with $\mathcal{B}_\infty$.) Then the following formula defines a measurable map $C[0,\infty)\times C[0,\infty) \to C[0,\infty)$:

$$(f,g) \mapsto h\,,$$
$$h(t) = \begin{cases} f(t) & \text{if } t \le T(f),\\ f(T(f)) + g(t-T(f)) - g(0) & \text{if } t \ge T(f). \end{cases}$$

The proof is left to the reader.

It follows that the distribution of h (if random...) depends only on the distribution of the pair (f,g), which boils down to the distribution of f and the distribution of g when f,g are independent.