1 Independent increments

1a Three convolution semigroups

A family $(\mu_t)_{t\in[0,\infty)}$ of probability measures on $\mathbb{R}$ is called a *convolution semigroup,* if

$$\mu_{s+t} = \mu_s * \mu_t \quad \text{for all } s,t \in [0,\infty)\,.$$

In terms of characteristic functions $\varphi_t(u) = \int \mathrm{e}^{\mathrm{i}ux}\,\mu_t(\mathrm{d}x)$ it means $\varphi_{s+t}(u) = \varphi_s(u)\varphi_t(u)$. In terms of densities $p_t(x) = \mathrm{d}\mu_t(x)/\mathrm{d}x$, if they exist, it means $p_{s+t}(x) = \int p_s(y)p_t(x-y)\,\mathrm{d}y$. Three examples are especially interesting.[1]

Normal distribution:

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{x^2}{2t}\Big)\,;$$
$$\varphi_t(u) = \exp\Big(-\frac12 tu^2\Big)\,.$$

Cauchy distribution:

$$p_t(x) = \frac{t}{\pi(t^2+x^2)}\,;$$
$$\varphi_t(u) = \exp(-t|u|)\,.$$

Lévy distribution:

$$p_t(x) = \frac{t}{\sqrt{2\pi}x^{3/2}} \exp\bigg(-\frac{t^2}{2x}\bigg)\,;$$
$$\varphi_t(u) = \exp\big(-t|u|^{1/2}(1-\mathrm{i}\operatorname{sgn} u)\big) = \exp\big(-t\sqrt{-2\mathrm{i}u}\big)\,.$$

Note that $p_t(x) = t^{-1/\alpha}p_1(xt^{-1/\alpha})$ where α (called the index) equals 2 for the normal case, 1 for Cauchy, and 1/2 for Lévy.

[1] These are the only cases of so-called stable laws in which the density is known explicitly. See also [1, Sect. 2.7].

1b Independent increments

Let $(\mu_t)_t$ be any one of the three convolution semigroups. We'll construct the corresponding probability measure on a space of functions. First, we consider the functions X on $\{0, 1, 2, \dots\}$ and require $X(0) = 0$ and

(1b1) the increments $X(t+1)-X(t)$ are independent, distributed μ_1 each.

That is,

(1b2)
$$\mathbb{E}\, f\bigl(X(0), X(1), \dots, X(n)\bigr) = \\ = \int \cdots \int f(0, x_1, \dots, x_n) p_1(x_1) p_1(x_2 - x_1) \dots p_1(x_n - x_{n-1})\, \mathrm{d}x_1 \dots \mathrm{d}x_n$$

for every n and every bounded continuous (or just Borel measurable) function $f : \mathbb{R}^{n+1} \to \mathbb{R}$. The distribution of X is the image of a product measure on $\mathbb{R}^\infty$.

Second, we consider the functions X on $\{0, 0.5, 1, 1.5, \dots\}$ and require $X(0) = 0$ and

(1b3)
the increments $X(t + 0.5) - X(t)$ are independent, distributed $\mu_{0.5}$ each.

Clearly, (1b3) implies (1b1). Continuing this way we get a consistent family of probability measures. Now, Kolmogorov's extension theorem (see [1, A.7]) gives us a probability measure on the space $\mathbb{R}^T$ of all functions on the set T of dyadic rationals $(k/2^n)$.

Alternatively, we may introduce conditional distributions (and densities):

$$\mathbb{E}\bigl(f(X_{n+0.5}) \,\big|\, X_n = x, X_{n+1} = z \bigr) = \int f(y)\, \underbrace{\frac{p_{0.5}(y - x) p_{0.5}(z - y)}{p_1(z - x)}}_{q(x,y,z)}$$

and let

$$\mathbb{E}\, f\bigl(X(0), X(0.5), \dots, X(n - 0.5), X(n)\bigr) = \\ \int \cdots \int \mathrm{d}x_1 \dots \mathrm{d}x_n p_1(x_1) p_1(x_2 - x_1) \dots p_1(x_n - x_{n-1}) \int \cdots \int \mathrm{d}x_{0.5} \dots \mathrm{d}x_{n-0.5} \cdot \\ \cdot q(x_0, x_{0.5}, x_1) \dots q(x_{n-1}, x_{n-0.5}, x_n) f(0, x_{0.5}, x_1, \dots, x_{n-0.5}, x_n)\,.$$

And similarly for all dyadic rationals. This way we avoid Kolmogorov's extension theorem. The needed distribution on $\mathbb{R}^T$ is the image of (say) the product Lebesgue measure on $[0, 1]^\infty$ under a rather explicit map...

We use both notations, $X(t)$ and X_t.

One way or another, we get a family $(X_t)_{t\in T}$ (T = dyadic rationals) of random variables $X_t : \Omega \to \mathbb{R}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ (be it $\mathbb{R}^T$ with the constructed measure, or the $[0,1]^\infty$, or just $[0,1]$ with Lebesgue measure; both $[0,1]^\infty$ and $[0,1]$ may be thought of as a countable collection of independent binary digits), and these X_t satisfy $X(0) = 0$ and for every n,

$$\text{the increments } X\Big(\frac{k+1}{2^n}\Big) - X\Big(\frac{k}{2^n}\Big) \text{ are independent, distributed } \mu_{2^{-n}} \text{ each.}$$

Now we forget the constructions and remember only these properties of the random variables X_t.

1b4 Exercise. (a) $X(t) - X(s)$ is distributed μ_{t-s} whenever $s, t \in T$, $s < t$;

(b) $X(t_1), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$ are independent whenever $t_1, \dots, t_n \in T$, $t_1 < \dots < t_n$.

Prove it.

1c Continuous time

We note that

$$X(s+t) - X(s) \sim X(t) \sim t^{1/\alpha} X(1)$$

(here '$\sim$' means: 'is distributed like') and conclude that the map $t \mapsto X_t$, from the dyadic rationals to $L_0(\Omega)$, is uniformly continuous. It can be extended to $[0,\infty)$ by continuity (since $L_0(\Omega)$ is complete).[1] If $t_n \to t$ fast enough then $X_{t_n} \to X_t$ a.s. (use the first Borel-Cantelli lemma…)[2]

1c1 Exercise. Generalize 1b4 from T = (dyadic rationals) to $T = [0,\infty)$.

Note that each random variable $X_t \in L_0(\Omega)$ is an equivelence class (rather than a function). Accordingly, we cannot introduce sample functions $t \mapsto X(t,\omega)$.[3] However, we may consider sample functions over dyadic rationals (or another countable subset of $[0,\infty)$).

We have three random processes with stationary independent increments: the Brownian motion, the Cauchy process, and the special Lévy process.[4]

[1] For the Brownian motion, the space $L_2(\Omega)$ may be used instead.

[2] Do you think that it holds whenever $t_n \to t$?

[3] Do you believe that this is possible?

[4] 'Special', since the term 'Lévy process' stands for an *arbitrary* process with stationary independent increments.

For now they are treated just as continuous maps $[0,\infty) \to L_0(\Omega)$,[1] however, in Sect. 1e they'll be upgraded to random functions.

Scaling relation: for every $c \in (0,\infty)$,

$$(X_{ct})_t \sim (c^{1/\alpha} X_t)_t \,; \qquad (\alpha = 2, 1, 0.5)$$

that is, these two processes are identically distributed, which means that for all n and $t_1, \dots, t_n \in [0,\infty)$,

$$\big(X(ct_1), \dots, X(ct_n) \big) \sim \big(c^{1/\alpha} X(t_1), \dots, c^{1/\alpha} X(t_n) \big) \,.$$

1c2 Exercise. Prove the scaling relation.

Equivalently we may say that the process

$$Y(t) = \mathrm{e}^{-t/\alpha} X(\mathrm{e}^t) \qquad \text{for } t \in \mathbb{R}$$

is stationary, that is, for every $s \in \mathbb{R}$,

$$(Y_{s+t})_t \sim (Y_t)_t \,.$$

1c3 Exercise. Prove that for every $s \in (0,\infty)$,

$$(X_t)_t \sim (X_{t+s} - X_s)_t \,.$$

1c4 Exercise. Prove the following relation (sometimes called time reversal)

$$(X_t)_{t \in [0,1]} \sim (X_1 - X_{1-t})_{t \in [0,1]} \,.$$

1d Bad behavior

Let $(X_t)_t$ be the Cauchy process.

1d1 Exercise. For every $c \in (0,\infty)$,

$$\mathbb{P}\big(|X(t)| > c \big) \sim \frac{2}{\pi} \frac{t}{c} \quad \text{as } t \to 0+ \,.$$

(This time '$\sim$' means: 'their ratio tends to 1'.)

Prove it.

[1] It does not mean continuous sample functions! A counterexample: $\Omega = [0,1]$, $T = [0,1]$, $X(t,\omega) = \operatorname{sgn}(t - \omega)$.

1d2 Exercise. For every $c \in (0, \infty)$,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\forall k = 1, \dots, n \quad \Big| X\Big(\frac{k}{n}\Big) - X\Big(\frac{k+1}{n}\Big) \Big| \le c \Big) \le \exp\Big(-\frac{2}{\pi c} \Big) .$$

Prove it.

1d3 Exercise. For every $c \in (0, \infty)$,

$$\mathbb{P}\Big(\lim_{\varepsilon \to 0+} \sup_{|s-t| \le \varepsilon} |X(s) - X(t)| \le c \Big) \le \exp\Big(-\frac{2}{\pi c} \Big) ;$$

here s, t run over dyadic rationals of $[0, 1]$.

Prove it.

1d4 Exercise. Almost surely, the sample function (of the Cauchy process) on dyadic rationals of $[0, 1]$ is not a uniformly continuous function (and therefore cannot be extended to $[0, 1]$ by continuity).

Prove it.

1d5 Exercise. Prove the same (as in 1d4) for the special Lévy process.[1]

Let $(X_t)_t$ be any one of the three processes.

1d6 Exercise. If $0 \le s < t < \infty$ and $T_1, T_2 \subset [s, t]$ are dense countable sets then

$$\sup_{T_1} X = \sup_{T_2} X \in [0, \infty] \quad \text{a.s.}$$

Prove it.

The same holds for $\inf X$. We introduce

$$\operatorname{Osc}(s, t) = \sup_T X - \inf_T X ;$$

it is an equivalence class of functions $\Omega \to [0, \infty]$;[2] the choice of a dense countable set $T \subset [s, t]$ does not matter.

1d7 Exercise. For every n,

$$\mathbb{P}\big(\operatorname{Osc}(0, 1) \le \varepsilon \big) = O(\varepsilon^n) \quad \text{as } \varepsilon \to 0+ \ .$$

Prove it.[3]

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[1] Could you do the same for the Brownian motion?

[2] Do you think that $\operatorname{Osc}(0, 1) < \infty$ a.s.? Wait for Sect. 1e.

[3] By the way, optimization in n gives a stronger result: $\mathbb{P}\big(\operatorname{Osc}(0, 1) \le \varepsilon\big) \le \exp(-\text{const} \cdot \varepsilon^{-\alpha})$.

1d8 Exercise. $\operatorname{Osc}(s,t) \sim \operatorname{Osc}(0,t-s) \sim t^{1/\alpha}\operatorname{Osc}(0,1)$.
Prove it.

1d9 Exercise. For every $\varepsilon > 0$ and $C < \infty$,

$$\mathbb{P}\Big(\exists k \in \{1,\dots,n\} \;\; n^{\frac{1}{\alpha}+\varepsilon}\operatorname{Osc}\big(\tfrac{k-1}{n},\tfrac{k}{n}\big) \le 1\Big) = O(n^{-C})$$

as $n \to \infty$.
Prove it.

The first Borel-Cantelli lemma gives

$$\forall k \in \{1,\dots,n\} \quad \operatorname{Osc}\big(\tfrac{k-1}{n},\tfrac{k}{n}\big) > n^{-\frac{1}{\alpha}-\varepsilon}$$

for all n large enough, a.s.

1d10 Exercise. For every $\varepsilon > 0$ and every dense countable set $T \subset [0,1]$,

$$\mathbb{P}\Big(\exists s \in (0,1)\; \exists x \in \mathbb{R} \;\; (t-s)^{-\frac{1}{\alpha}-\varepsilon}|X_t - x| \to 0 \;\text{ as } t \to s+, t \in T\Big) = 0.$$

Prove it.

A wonder: the set is the union of a continuum of sets, and nevertheless we are able to prove that it is negligible.

It follows immediately that sample functions of the Brownian motion, if they exist, must be nowhere differentiable. This claim is much stronger than a.s. nondifferentiability at every point separately.

1e Good behavior

1e1 Exercise. Let $(X_t)_t$ be the special Lévy process, and $T \subset [0,1]$ a dense countable set. Then $X(t) \to 0$ a.s. as $t \to 0$, $t \in T$.
Prove it.[1]

Let $(X_t)_t$ be any one of the three processes.
We introduce

$$M(0,1) = \sup_T |X|\,,$$

where $T \subset [0,1]$ is a dense countable set (no matter which one, recall 1d6); once again, $M(0,1)$ is an equivalence class of functions $\Omega \to [0,\infty]$.

[1](a) Does it contradict 1d5? (b) Do you think that 1e1 holds also for the Brownian motion? For the Cauchy process?

1e2 Lemma. For every $c \in (0, \infty)$,

$$\mathbb{P}\big(M(0,1) > c\big) \le 2\mathbb{P}\big(|X_1| > c\big)\,.$$

A wonder: the union of many events is roughly of the same probability as every one separately!

Proof. First, $L_n \uparrow M(0,1)$ a.s., where

$$L_n = \max\Bigg(\Big|X\Big(\frac{1}{2^n}\Big)\Big|, \dots, \Big|X\Big(\frac{2^n-1}{2^n}\Big)\Big|, |X(1)|\Bigg)\,.$$

Therefore $\mathbb{P}\big(L_n > c\big) \uparrow \mathbb{P}\big(M(0,1) > c\big)$;[1] it is sufficient to prove that $\mathbb{P}\big(L_n > c\big) \le 2\mathbb{P}\big(|X_1| > c\big)$ for all n.

We define measurable $A_k \subset \Omega$ for $k = 1, \dots, 2^n$ by

$$A_k = \Bigg\{\omega : \Big|X\Big(\frac{1}{2^n}\Big)\Big| \le c, \dots, \Big|X\Big(\frac{k-1}{2^n}\Big)\Big| \le c, \Big|X\Big(\frac{k}{2^n}\Big)\Big| > c\Bigg\}$$

and get

$$\mathbb{P}\big(L_n > c\big) = \mathbb{P}\big(A_1 \cup \dots \cup A_{2^n}\big) = \mathbb{P}\big(A_1\big) + \dots + \mathbb{P}\big(A_{2^n}\big)$$

(think, why). We introduce $B = \{\omega : |X_1| > c\}$ and note that

$$\mathbb{P}\big(A_k \cap B\big) \ge \frac{1}{2}\mathbb{P}\big(A_k\big) \quad \text{for all } k$$

(think, why). Thus, $\sum_k \mathbb{P}\big(A_k\big) \le 2\sum_k \mathbb{P}\big(A_k \cap B\big) \le 2\mathbb{P}\big(B\big)$. □

It follows that $M(0,1) < \infty$ a.s.

We note that $\operatorname{Osc}(0,1) \le 2M(0,1)$ and conclude that

$$\operatorname{Osc}(s,t) < \infty \quad \text{a.s.}$$

1e3 Exercise. For every $\varepsilon \in (0, \infty)$,

$$\mathbb{P}\big(\operatorname{Osc}(0,t) > \varepsilon\big) = O(t) \quad \text{as } t \to 0+\,, \tag{1e4}$$

and for the Brownian motion, moreover,

$$\mathbb{P}\big(\operatorname{Osc}(0,t) > \varepsilon\big) = o(t) \quad \text{as } t \to 0+\,, \tag{1e5}$$

Prove it.

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[1] Do you think that also $\mathbb{P}\big(L_n \ge c\big) \uparrow \mathbb{P}\big(M(0,1) \ge c\big)$?

1e6 Exercise. For the Brownian motion,

$$\max_{k=1,\dots,2^n-1} \operatorname{Osc}\Big(\frac{k-1}{2^n},\frac{k+1}{2^n}\Big) \to 0 \quad \text{a.s.}$$

Prove it.

It follows that almost surely, the sample function of the Brownian motion on dyadic rationals of $[0,1]$ (or another dense countable $T \subset [0,1]$) is a uniformly continuous function (in contrast to the other two processes, recall 1d4, 1d5), and therefore it can be extended to $[0,1]$ by continuity. On the whole $[0,\infty)$ it need not be uniformly continuous, but still, can be extended by continuity.

Doing so, we upgrade the family $(X_t)_t$ of equivalence classes to a random function, — an equivalence class of maps $\Omega \to \mathbb{R}^{[0,\infty)}$. The latter object is what is called the Brownian motion, and denoted $(B_t)_t$.

$$\boxed{\text{Brownian sample functions are continuous.}}$$

Taking 1d10 into account we conclude that a Brownian sample function is everywhere continuous, but nowhere differentiable. Therefore, such functions exist!

For the special Lévy process the situation is different. Its sample functions on T are increasing, therefore they have (finite) left limits and right limits,

$$X_-(t) = \lim_{\substack{s\to t- \\ s\in T}} X(s) \quad \text{for } t\in(0,\infty)\,, \qquad X_+(t) = \lim_{\substack{s\to t+ \\ s\in T}} X(s) \quad \text{for } t\in[0,\infty)\,.$$

1e7 Exercise. For the special Lévy process,

$$\forall t \in (0,\infty)\;\; \mathbb{P}\big(X_-(t) = X_+(t)\big) = 1\,, \tag{1e8}$$

$$\mathbb{P}\big(\forall t \in (0,\infty)\;\; X_-(t) = X_+(t)\big) = 0\,. \tag{1e9}$$

Prove it.

It follows that $X_-(t)$, $X_+(t)$ and $X(t)$ are in the same equivalence class. We have at least two reasonable ways of upgrading the family of equivalence classes to a random function: by left continuity, or by right continuity. Also $(X_-(t)+X_+(t))/2$ is an option... Traditionally, one prefers right continuity.

$$\boxed{\text{Sample functions of the special Lévy process are r.c.l.l.}}$$

Here 'r.c.l.l.' means: right continuous, having (finite) left limit (at every point). Jumps exist, but a given point is a.s. not a jump; in other words: no fixed jumps.

We turn to the Cauchy process. Still, (1e8) and (1e9) hold. It means: no fixed discontinuities. However, are there discontinuities worse than jumps? In other words: do the left and right limits exist?

We would like to discuss the probability of the event

$$\forall t \in (0,1) \quad X_-(t) \text{ and } X_+(t) \text{ exist}\,, \tag{1e10}$$

however, it is not evident that this is really an event, that is, the set of such ω is measurable! Thus we turn to the event

$$\begin{gathered}\exists \varepsilon > 0\ \forall n\ \exists t_1, \dots, t_n \in T \\ \big(t_1 < t_2 < \dots < t_n,\ |X(t_1) - X(t_2)| \ge \varepsilon, \dots, |X(t_{n-1}) - X(t_n)| \ge \varepsilon \big)\,.\end{gathered} \tag{1e11}$$

(As before, $T \subset [0,1]$ is a given dense countable set, and $(X_t)_t$ is the Cauchy process.) This is a measurable subset of Ω, and it contains the complement of (1e10) (think, why).[1] We'll prove that (1e11) is of probability 0, and therefore (1e10) holds a.s.

In order to prove that (1e11) is of probability 0 it is sufficient to do it for every ε separately (think, why). Let ε be given. For a finite or countable set $T \subset [0,\infty)$ we consider the event

$$\begin{aligned}A_n(T) = \{\omega : {}&\exists t_1, \dots, t_n \in T \\ &\big(t_1 < t_2 < \dots < t_n, |X(t_1) - X(t_2)| \ge \varepsilon, \dots, |X(t_{n-1}) - X(t_n)| \ge \varepsilon \big)\,.\end{aligned}$$

It is sufficient to prove that $\mathbb{P}\big(A_n(T)\big) \to 0$ as $n \to \infty$.

It is easy to estimate $\mathbb{P}\big(|X(t_1) - X(t_2)| \ge \varepsilon, \dots, |X(t_{n-1}) - X(t_n)| \ge \varepsilon\big)$ when $t_1 < t_2 < \dots < t_n$ are given (think, how). But we need the union over all such $t_1, \dots, t_n$! We'll do it in the spirit of 1e2.

1e12 Lemma. [2] Let $T \subset [0,1]$ be finite, then

$$\mathbb{P}\big(A_{n+1}(T)\big) \le \mathbb{P}\big(\operatorname{Osc}(0,1) \ge \varepsilon\big) \cdot \mathbb{P}\big(A_n(T)\big)$$

for all n.

Proof. We introduce

$$C(t) = \{\omega : \operatorname{Osc}(0,t) \ge \varepsilon, \text{ and } \forall s \in T \cap [0,t) \quad \operatorname{Osc}(0,s) < \varepsilon\}$$

and get, on one hand,

$$\sum_{t\in T} \mathbb{P}\big(C(t)\big) = \mathbb{P}\Big(\bigcup_{t\in T} C(t)\Big) \le \mathbb{P}\big(\operatorname{Osc}(0,1) \ge \varepsilon\big)$$

[1] Moreover, it is equal to that complement.

[2] See also [2, Sect. 2.2, Lemma 21].

and on the other hand (splitting at $t_2 \dots$)

$$\mathbb{P}\big(A_{n+1}(T)\big) \le \mathbb{P}\Big(\exists t \in T \ \ \Big(\operatorname{Osc}(0,t) \ge \varepsilon, \text{ and } A_n(T \cap [t,1])\Big)\Big) =$$
$$= \mathbb{P}\Big(\bigcup_{t\in T}\big(C(t) \cap A_n(T \cap [t,1])\big)\Big) = \sum_{t\in T} \mathbb{P}\big(C(t) \cap A_n(T \cap [t,1])\big) =$$
$$= \sum_{t\in T} \mathbb{P}\big(C(t)\big) \cdot \mathbb{P}\big(A_n(T \cap [t,1])\big) \le \mathbb{P}\big(A_n(T)\big) \sum_{t\in T} \mathbb{P}\big(C(t)\big) \le$$
$$\le \mathbb{P}\big(A_n(T)\big) \cdot \mathbb{P}\big(\operatorname{Osc}(0,1) \ge \varepsilon\big) .$$

□

The same inequality for countable T follows, since $T_i \uparrow T$ implies $A_n(T_i) \uparrow A_n(T)$. Similarly,

$$\mathbb{P}\big(A_{n+1}(T)\big) \le \mathbb{P}\big(\operatorname{Osc}(0,t) \ge \varepsilon\big) \cdot \mathbb{P}\big(A_n(T)\big)$$

for $T \subset [0,t]$. For t small enough we have $\mathbb{P}\big(\operatorname{Osc}(0,t) \ge \varepsilon\big) < 1$ and therefore $\mathbb{P}\big(A_n(T)\big) \to 0$ as $n \to \infty$. Taking into account that

$$A_{2n}(T) \subset A_n(T \cap [0,t]) \cup A_n(T \cap [t,\infty))$$

we pass from t to $2t$, and so on. Thus, (1e10) holds a.s., and the situation for the Cauchy process is the same as for the special Lévy process.

$\boxed{\text{Sample functions of the Cauchy process are r.c.l.l.}}$

It follows that jumps of size $\ge \varepsilon$ are a locally finite set (which can be deduced also from (1e4)). However, the set of all jumps is dense.

1f Hints to exercises

1c1: Random variables U, V are independent if and only if $\mathbb{E}\,\big(f(U)g(V)\big) = \big(\mathbb{E}\,f(U)\big)\big(\mathbb{E}\,g(V)\big)$ for all bounded continuous $f, g : \mathbb{R} \to \mathbb{R}$.

1d1: Estimate the integral of the density.

1d6: $X(t) \le \sup_T X$ a.s.

1d7: $\mathbb{P}\big(\operatorname{Osc}(0,1) \le \varepsilon\big) \le \mathbb{P}\big(|X(\frac1n)| \le \varepsilon,\ |X(\frac2n) - X(\frac1n)| \le \varepsilon,\ \dots,\ |X(1) - X(\frac{n-1}{n})| \le \varepsilon\big)$; estimate the integral of the density.

1d9: $\mathbb{P}\big(A_1 \cup \dots \cup A_n\big) \le \mathbb{P}\big(A_1\big) + \dots + \mathbb{P}\big(A_n\big)$.

1d10: $s \in [\frac{k-2}{n}, \frac{k-1}{n}]$, $t \in [\frac{k-1}{n}, \frac{k}{n}]$, $M\big(\frac{k-1}{n}, \frac{k}{n}\big) > n^{-\frac{1}{\alpha} - \frac{\varepsilon}{2}}$.

1e1: The limit exists by monotonicity.

1e3: Use 1e2; estimate the integral of the density.

1e6: The limit exists by monotonicity; use 1e5.

1e7: (a) $\operatorname{Osc}(t-\varepsilon, t+\varepsilon) \to 0$ a.s. as $\varepsilon \to 0+$; (b) use 1d5.

References

[1] R. Durrett, *Probability: theory and examples,* 1996.

[2] A.V. Skorohod, *Random processes with independent increments,* 1991. (Transl. from Russian.)

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