4 Brownian martingales

4a Heat equation appears

Recall 3a5: $\mathbb{E}\, B^2(T) = \mathbb{E}\, T$ for every stopping time T such that (say) $T \le 1$ a.s. This is a manifestation of the martingale property of the process

$$M(t) = B^2(t) - t \,,$$

as explained below. Here is a solution of 3a5 (hopefully, not new to you). Using 2f8 we have[1]

$$\begin{aligned} 0 = \mathbb{E}\,(B^2(1) - 1) &= \iint \bigl(Y^2(1)(\omega_1,\omega_2) - 1 \bigr) \, P(\mathrm{d}\omega_1) P(\mathrm{d}\omega_2) = \\ &= \int P(\mathrm{d}\omega_1) \int P(\mathrm{d}\omega_2) \Bigl(\bigl((B(T(\omega_1))(\omega_1) + B(1-T(\omega_1))(\omega_2) \bigr)^2 - 1 \Bigr) = \\ &= \int P(\mathrm{d}\omega_1) f\bigl(T(\omega_1), B(T(\omega_1))(\omega_1) \bigr) = \mathbb{E}\, g(T, B(T)) = \mathbb{E}\,(B^2(T) - T) \,, \end{aligned}$$

where

$$g(t,x) = \int P(\mathrm{d}\omega_2) \bigl((x + B(1-t)(\omega_2))^2 - 1 \bigr) = \mathbb{E}\, \bigl((x + B(1-t))^2 - 1 \bigr) = x^2 - t \,.$$

The relevant property of the function $f(t,x) = x^2 - t$ is $\mathbb{E}\, f(1, x + B(1-t)) = f(t,x)$. More generally,[2]

$$\text{(4a1)} \quad \mathbb{E}\, f\bigl(s+t, x+B(t)\bigr) = f(s,x) \,, \quad \text{that is,} \quad \int f(s+t, x+y) p_t(y) \, \mathrm{d}y = f(s,x) \,.$$

[1] Compare it with the proof of (2a7).

[2] As before, $p_t(x) = (2\pi t)^{-1/2} \exp(-\frac{x^2}{2t})$.

Three examples of such functions:

(4a2)
$$
\begin{aligned}
f(t,x) &= x\,,\\
f(t,x) &= x^2 - t\,,\\
f(t,x) &= x^3 - 3tx
\end{aligned}
$$

(check it). We define new functions f_{+t} for $t \in [0,\infty)$ by[1]

(4a3)
$$
f_{+t}(s,x) = \mathbb{E}\, f\big(s+t, x+B(t)\big) = \int f(s+t,x+y)p_t(y)\,\mathrm{d}y
$$

and note that $(f_{+t})_{+u} = f_{+(t+u)}$ (think, why). Now, the idea is simple and natural. We have a dynamics in (some) space of functions, and (4a1) means that f is a fixed point,

$$
f_{+t} = f \quad \text{for all } t \ge 0\,,
$$

that is, the speed vanishes at f,

$$
\frac{1}{\varepsilon}(f_{+\varepsilon} - f) \to 0 \quad \text{as } \varepsilon \to 0+\ .
$$

Denoting for convenience $\frac{\partial^{i+j}}{\partial s^i \partial x^j} f(s,x)$ by $f_{i,j}(s,x)$ we have for small t, y

$$
f(s+t,x+y) \approx f(s,x) + f_{1,0}(s,x)t + f_{0,1}(s,x)y + \frac{1}{2} f_{0,2}(s,x)y^2\,;
$$

$$
f_{+\varepsilon}(s,x) = \mathbb{E}\, f\big(s+\varepsilon, x+B(\varepsilon)\big) \approx f(s,x) + f_{1,0}(s,x)\varepsilon + \frac{1}{2} f_{0,2}(s,x) \underbrace{\mathbb{E}\, B^2(\varepsilon)}_{=\varepsilon}\,;
$$

$$
\frac{1}{\varepsilon}(f_{+\varepsilon} - f) \to f_{1,0} + \frac{1}{2} f_{0,2}\,.
$$

No one of the higher terms contributes (think, why). Thus, we guess that (4a1) is equivalent to a partial differential equation (PDE) well-known as the heat equation:[2]

(4a4)
$$
\Big(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \Big) f(t,x) = 0\,.
$$

The question is, how to prove it, and what to require of f.

---

[1] Assuming integrability. Of course, p_t does not work for $t = 0$.

[2] Or rather, time reversed heat equation with coefficient 1/2; the standard heat equation contains $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$.

4a5 Lemma. Let $f : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ be a continuous function such that the derivatives $f_{i,j}$ exist and are continuous for $(i,j) \in \{(1,0),(0,2)\}$. Assume that[1]

$$\frac{1}{x^2}\ln^+|f_{i,j}(t,x)| \to 0 \quad \text{as } x \to \pm\infty \tag{4a6}$$

for every $t \in (0,\infty)$, $(i,j) \in \{(0,0),(1,0),(0,2)\}$, and moreover, it holds uniformly in $t \in [a,b]$ whenever $0 < a < b < \infty$. Then f_{+t} is well-defined (by (4a3)) for all $t \in (0,\infty)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{+t}(s,x) = \int \Big(f_{1,0}(s+t,x+y) + \frac{1}{2} f_{0,2}(s+t,x+y)\Big) p_t(y)\,\mathrm{d}y$$

for all $t \in (0,\infty)$ and $(s,x) \in (0,\infty)\times\mathbb{R}$. Both sides are claimed to be well-defined (the derivative in the left-hand side and the integral in the right-hand side).

4a7 Exercise. For every twice continuously differentiable function $g : [x-\varepsilon, x+\varepsilon] \to \mathbb{R}$,

$$\min_{[x-\varepsilon,x+\varepsilon]} g''(\cdot) \le \frac{g(x-\varepsilon)-2g(x)+g(x+\varepsilon)}{\varepsilon^2} \le \max_{[x-\varepsilon,x+\varepsilon]} g''(\cdot)\,.$$

Prove it.

Proof of 4a5. First, f_{+t} is well-defined due to (4a6) for $(i,j)=(0,0)$.

Second, without loss of generality we assume that $s=0$, $x=0$ (since the shifted function $(s_1,x_1) \mapsto f(s+s_1,x+x_1)$ satisfies all the conditions imposed on f).

Right derivative is considered below; left derivative, treated similarly, is left to the reader.

We have for every $\varepsilon > 0$

$$f_{+t}(0,0) = \int f(t,y)p_t(y)\,\mathrm{d}y\,;$$

$$f_{+(t+\varepsilon)}(0,0) = \int f(t+\varepsilon,y)p_{t+\varepsilon}(y)\,\mathrm{d}y =$$
$$= \int \mathrm{d}y\, p_t(y) \int \mathrm{d}z\, p_\varepsilon(z) f(t+\varepsilon,y+z) = \int \mathrm{d}y\, p_t(y) \int \mathrm{d}z\, p_1(z) f(t+\varepsilon,y+z\sqrt{\varepsilon}) =$$
$$= \int \mathrm{d}y\, p_t(y) \int \mathrm{d}z\, p_1(z) \frac{f(t+\varepsilon,y-z\sqrt{\varepsilon})+f(t+\varepsilon,y+z\sqrt{\varepsilon})}{2}\,;$$

[1] Here $\ln^+ a = \max(0,\ln a)$.

$$\frac{f_{+(t+\varepsilon)}(0,0) - f_{+t}(0,0)}{\varepsilon} = \\ = \int \mathrm{d}y\, p_t(y) \int \mathrm{d}z\, p_1(z) \frac{f(t+\varepsilon, y - z\sqrt{\varepsilon}) - 2f(t,y) + f(t+\varepsilon, y+z\sqrt{\varepsilon})}{2\varepsilon} \, .$$

By 4a7 and continuity of $f_{0,2}$,

$$\frac{f(t+\varepsilon, y - z\sqrt{\varepsilon}) - 2f(t+\varepsilon,y) + f(t+\varepsilon, y+z\sqrt{\varepsilon})}{2\varepsilon} \to \frac{z^2}{2} f_{0,2}(t,y) \quad \text{as } \varepsilon \to 0+ \, .$$

Taking into account that

$$\frac{f(t+\varepsilon, y) - f(t,y)}{\varepsilon} \to f_{1,0}(t,y)$$

we get

$$\frac{f(t+\varepsilon, y - z\sqrt{\varepsilon}) - 2f(t,y) + f(t+\varepsilon, y+z\sqrt{\varepsilon})}{2\varepsilon} \to f_{1,0}(t,y) + \frac{z^2}{2} f_{0,2}(t,y)$$

as $\varepsilon \to 0+$. Now we need an integrable majorant. Using 4a7 again,

$$\left| \frac{f(t+\varepsilon, y - z\sqrt{\varepsilon}) - 2f(t+\varepsilon,y) + f(t+\varepsilon, y+z\sqrt{\varepsilon})}{2\varepsilon} \right| \le \\ \max_{[y-|z|\sqrt{\varepsilon}, y+|z|\sqrt{\varepsilon}]} |f_{0,2}(t+\varepsilon, \cdot)| \le C(\delta) \exp\bigl(\delta(|y|+|z|\sqrt{\varepsilon})^2\bigr) \le C(\delta) \exp\bigl(2\delta(y^2+z^2\varepsilon)\bigr)$$

by (4a6) for $f_{0,2}$ (locally uniform in $t\dots$); any $\delta > 0$ may be chosen. Also,

$$\left| \frac{f(t+\varepsilon, y) - f(t,y)}{\varepsilon} \right| \le \max_{[t,t+\varepsilon]} |f_{1,0}(\cdot, y)| \le C(\delta) \exp(\delta y^2)$$

by (4a6) for $f_{1,0}$ (locally uniform in t). We have a majorant

$$C(\delta) \exp\bigl(2\delta(y^2 + z^2\varepsilon)\bigr) p_t(y) p_1(z) \, ,$$

integrable if δ is small enough (namely, $2\delta < \frac{1}{2t}$ and $2\delta\varepsilon < 1/2$). By the dominated convergence theorem (applied to $\iint \mathrm{d}y \mathrm{d}z \dots$),

$$\frac{f_{+(t+\varepsilon)}(0,0) - f_{+t}(0,0)}{\varepsilon} \to \int \mathrm{d}y\, p_t(y) \int \mathrm{d}z\, p_1(z) \Bigl(f_{1,0}(t,y) + \frac{z^2}{2} f_{0,2}(t,y) \Bigr) = \\ = \int \mathrm{d}y\, p_t(y) \Bigl(f_{1,0}(t,y) + \frac{1}{2} f_{0,2}(t,y) \Bigr) \quad \text{as } \varepsilon \to 0+ \, .$$

$\square$

4a8 Exercise. Consider in detail the other case: left derivative.

See also [1], Sect. 7.5, Exercise 5.5.

4a9 Proposition. Condition (4a1) is equivalent to the PDE (4a4) for every function satisfying the conditions of Lemma 4a5.

Proof. Follows from 4a5, since $f_{+\varepsilon} \to f$ as $\varepsilon \to 0+$. □

4a10 Proposition. $\mathbb{E}\, f\big(T, B(T)\big) = f(0,0)$ for every function f satisfying the conditions of Lemma 4a5 and the PDE (4a4), and every stopping time T such that $\exists t\ \mathbb{P}\big(T \le t\big) = 1$.

The proof given for $f(t,x) = x^2 - t$ in the beginning of 4a generalizes immediately.

4a11 Exercise. For every polynomial P on $\mathbb{R}$ the following polynomial f on $\mathbb{R}^2$ satisfies (4a1):

$$f(t,x) = \sum_k (-1)^k \frac{2^k}{(2k)!} P^{(k)}(t) x^{2k} \,.$$

Prove it.

Now we can continue (4a2) a little:

$$\begin{aligned} f(t,x) &= x^2 - t \,, & P(t) &= -t \,, \\ f(t,x) &= x^4 - 6tx^2 + 3t^2 \,, & P(t) &= 3t^2 \,. \end{aligned}$$

4a12 Exercise. [1] $\operatorname{Var} T = 2/3$ for $T = \min\{t : |B(t)| = 1\}$.
Prove it. (Warning: be careful with $t \to \infty$.)

An astonishing counterexample was found by Tychonoff[2],[3]: let

$$P(t) = \begin{cases} \exp\big(-(1-t)^{-2}\big) & \text{for } t \in [0,1), \\ 0 & \text{for } t \in [1,\infty) \end{cases}$$

(not a polynomial, of course, but a non-analytic infinitely differentiable function), then the formula given in 4a11 produces a power series convergent for

[1] See [1], Sect. 7.5, Theorem (5.9).
[2] A.N. Tychonoff, Matem. Sbornik **32** (1935), 199–216.
[3] See (1.18)–(1.24) on page 212 in: F. John, "Partial differential equations", Springer (fourth edition).

all x (and all t) to an infinitely differentiable function that satisfies the PDE (4a4) but violates (4a1).

Trying $f(t,x) = \exp(at+bx)$ we get $f_{1,0} = af$ and $f_{0,2} = b^2 f$, thus, (4a1) is satisfied if and only if $a + 0.5b^2 = 0$;

$$f(t,x) = \mathrm{e}^{\lambda x}\mathrm{e}^{-\lambda^2 t/2}\,.$$

Also functions

$$\frac{f(t,x)+f(t,-x)}{2} = \mathrm{e}^{-\lambda^2 t/2}\cosh\lambda x\,,$$
$$\frac{f(t,x)-f(t,-x)}{2} = \mathrm{e}^{-\lambda^2 t/2}\sinh\lambda x$$

satisfy (4a1). Replacing λ with $\mathrm{i}\lambda$ we get functions

$$f(t,x) = \mathrm{e}^{\lambda^2 t/2}\cos\lambda x\,,$$
$$f(t,x) = \mathrm{e}^{\lambda^2 t/2}\sin\lambda x$$

satisfying (4a1).

4a13 Exercise. [1] Let $T = \min\{t : |B(t)| = 1\}$, then

$$\mathbb{E}\,\mathrm{e}^{\lambda T} = \begin{cases} \frac{1}{\cosh\sqrt{2|\lambda|}} & \text{for } -\infty < \lambda \le 0, \\ \frac{1}{\cos\sqrt{2\lambda}} & \text{for } 0 \le \lambda < \pi^2/8, \\ \infty & \text{for } \lambda \ge \pi^2/8. \end{cases}$$

Prove it. Also, give a new proof of 3a7.

4b Conditioning and martingales

Conditioning is simple in two frameworks: discrete probability, and densities. However, conditioning of a Brownian motion on its past goes far beyond these two frameworks. The clue is, the 'restart' introduced in Sect. 2:

$$X(t)(\omega_1,\omega_2) = \begin{cases} B(t)(\omega_1) & \text{if } t \le T(\omega_1), \\ B(T(\omega_1))(\omega_1) + B(t-T(\omega_1))(\omega_2) & \text{if } t \ge T(\omega_1). \end{cases}$$

Here T is a stopping time (as defined by 2f5) and of course, $T(\omega)$ means $T\big(B(\cdot)(\omega)\big)$. Let us write X in a shorter form:

$$\text{(4b1)} \qquad X(\cdot)(\omega_1,\omega_2) = B(\cdot)(\omega_1) \bigsqcup^{T(\omega_1)} B(\cdot)(\omega_2)\,,$$

[1] See also [1], Sect. 7.5, Th. (5.7).

where $f \overset{a}{\sqcup} g$ is defined for $f, g \in C[0,\infty)$ and $a \in [0,\infty)$ by

$$f \bigsqcup^{a} g = h \in C[0,\infty)\,,$$

$$h(t) = \begin{cases} f(t) & \text{if } t \le a, \\ f(a) + g(t-a) - g(0) & \text{if } t \ge a. \end{cases}$$

Not only the map $(f,g) \mapsto f \overset{a}{\sqcup} g$ is Borel measurable for each a, but also

$$(f,a,g) \mapsto f \bigsqcup^{a} g$$

is a Borel measurable map $C[0,\infty) \times \mathbb{R} \times C[0,\infty) \to C[0,\infty)$, and therefore $(f,g) \mapsto f \overset{T(f)}{\sqcup} g$ is a Borel measurable map $C[0,\infty) \times C[0,\infty) \to C[0,\infty)$.[1]

4b2 Definition. The conditional distribution of $B(\cdot)$ given $B(\cdot)|_{[0,a]} = f|_{[0,a]}$ is the distribution of $f \overset{a}{\sqcup} B$.

Some Borel functions $\varphi : C[0,\infty) \to \mathbb{R}$ are such that $\varphi\big(f \overset{T(f)}{\sqcup} g\big)$ depends on f only (not g). Some of these functions are indicators, $\varphi : C[0,\infty) \to \{0,1\}$. The corresponding sets are, by definition, the sub-σ-field $\mathcal{B}_T \subset \mathcal{B}_\infty$. A Borel function $\varphi : C[0,\infty) \to \mathbb{R}$ is $\mathcal{B}_T$-measurable if and only if $\varphi\big(f \overset{T(f)}{\sqcup} g\big)$ depends on f only (think, why).

Events of the form $\{B(\cdot) \in G\}$ for $G \in \mathcal{B}_T$ are, by definition, the sub-σ-field $\mathcal{F}_T$ on Ω. Especially, the sub-σ-field $\mathcal{F}_\infty$ on Ω generated by the Brownian motion consists of the events of the form $\{B(\cdot) \in G\}$ for $G \in \mathcal{B}_\infty$.[2]

Conditional probability of an event of $\mathcal{F}_\infty$ given $\mathcal{F}_T$ is (by definition) the random variable

$$\text{(4b3)} \qquad \mathbb{P}\big(B(\cdot) \in G \,\big|\, \mathcal{F}_T\big)(\omega_1) = \mathbb{P}\left(\{\omega_2 : B(\cdot)(\omega_1) \bigsqcup^{T(\omega_1)} B(\cdot)(\omega_2) \in G\}\right).$$

In other words, it is the probability according to the conditional distribution of $B(\cdot)$ given $B(\cdot)|_{[0,T]}$. If $G \in \mathcal{B}_T$ then $\mathbb{P}\big(B(\cdot) \in G \,\big|\, \mathcal{F}_T\big) = \mathbb{1}_G$ (as it should be).

By the Fubini theorem,

$$\mathbb{E}\left(\mathbb{P}\big(B(\cdot) \in G \,\big|\, \mathcal{F}_T\big)\right) = \mathbb{P}\big(X(\cdot) \in G\big)\,,$$

[1] Recall Lemma B in Correction to 2f.

[2] Often $\mathcal{F}_\infty = \mathcal{F}$; always $\mathcal{F}_\infty \subset \mathcal{F}$; and sometimes $\mathcal{F}_\infty \ne \mathcal{F}$, recall 3c ($\Omega \times \{0,1,\dots,n\}^\infty$).

X being defined by (4b1). Thus, the strong Markov property turns into the total probability formula

$$\mathbb{P}\big(B(\cdot) \in G\big) = \mathbb{E}\left(\mathbb{P}\big(B(\cdot) \in G \,\big|\, \mathcal{F}_T\big)\right) \quad \text{for } G \in \mathcal{B}_\infty \,.$$

In other words,

$$\mathbb{P}\big(A\big) = \mathbb{E}\left(\mathbb{P}\big(A \,\big|\, \mathcal{F}_T\big)\right) \quad \text{for } A \in \mathcal{F}_\infty \,.$$

You may be astonished if you are acquainted with the general theory of conditioning. In that framework the total probability formula holds for every sub-σ-field, irrespective of any Markov property!

In that framework, however, conditional probabilities are defined as to satisfy this formula, for each sub-σ-field separately.[1] In contrast, we define them constructively by (4b3). In our definitions, the conditional distribution of $B(\cdot)$ given $B(\cdot)|_{[0,T]}$ at ω depends on $T(\omega)$ and the path on $[0, T(\omega)]$, not on the choice of T (as far as $T(\omega)$ is fixed).

The space $L_1(\mathcal{F}_\infty) = L_1(\Omega, \mathcal{F}_\infty, P)$ consists of (equivalence classes of) random variables of the form $\varphi\big(B(\cdot)\big)$ where $\varphi : C[0,\infty) \to \mathbb{R}$ is a Borel function such that $\mathbb{E}\,|\varphi\big(B(\cdot)\big)| < \infty$.

Conditional expectation of a randon variable of $L_1(\mathcal{F}_\infty)$ given $\mathcal{F}_T$ is (by definition) the random variable

$$\text{(4b4)} \qquad \mathbb{E}\big(\varphi(B(\cdot)) \,\big|\, \mathcal{F}_T\big)(\omega_1) = \int \varphi\Big(B(\cdot)(\omega_1) \bigsqcup^{T(\omega_1)} B(\cdot)(\omega_2)\Big) P(\mathrm{d}\omega_2) \,.$$

In other words, it is the expectation according to the conditional distribution of $B(\cdot)$ given $B(\cdot)|_{[0,T]}$.

4b5 Exercise. Prove the total expectation formula:

$$\mathbb{E}\,\varphi(B(\cdot)) = \mathbb{E}\left(\mathbb{E}\big(\varphi(B(\cdot)) \,\big|\, \mathcal{F}_T\big)\right)$$

for $\varphi(B(\cdot)) \in L_1(\mathcal{F}_\infty)$.

In other words:

$$\mathbb{E}\,X = \mathbb{E}\left(\mathbb{E}\big(X \,\big|\, \mathcal{F}_T\big)\right) \quad \text{for} X \in L_1(\mathcal{F}_\infty) \,.$$

[1] An example in the framework of densities: the conditional density of X given $Y = 0$ is proportional to $p_{X,Y}(x, 0)$, but the conditional density of X given $Y/X = 0$ is proportional to $|x| p_{X,Y}(x, 0)$.

4b6 Exercise. For all $X \in L_\infty(\mathcal{F}_T)$, $Y \in L_1(\mathcal{F}_\infty)$,

$$\mathbb{E}\big(XY \,\big|\, \mathcal{F}_T\big) = X\mathbb{E}\big(Y \,\big|\, \mathcal{F}_T\big)\,.$$

Prove it.

4b7 Definition. A *Brownian martingale*[1] is a family $(M_t)_{t\in[0,\infty)}$ of $M_t \in L_1(\mathcal{F}_t)$ such that

$$\mathbb{E}\big(M_t \,\big|\, \mathcal{F}_s\big) = M_s \quad \text{a.s. for } 0 \le s \le t < \infty\,.$$

(I write M_t and $M(t)$ interchangingly.)

Examples of Brownian martingales:

$$M(t) = B(t)\,, \qquad M(t) = B^2(t) - t\,,$$
$$M(t) = B^3(t) - 3tB(t)\,, \qquad M(t) = B^4(t) - 6tB^2(t) + 3t^2\,,$$
$$M(t) = \mathrm{e}^{-\lambda^2 t/2} \exp \lambda B(t)\,,$$
$$M(t) = \mathrm{e}^{-\lambda^2 t/2} \cosh \lambda B(t)\,, \qquad M(t) = \mathrm{e}^{-\lambda^2 t/2} \sinh \lambda B(t)\,,$$
$$M(t) = \mathrm{e}^{\lambda^2 t/2} \cos \lambda B(t)\,, \qquad M(t) = \mathrm{e}^{\lambda^2 t/2} \sin \lambda B(t)$$

and, more generally,

$$M(t) = f\big(t, B(t)\big) \tag{4b8}$$

where f satisfies the conditions of Lemma 4a5 and the PDE (4a4). And, of course, the process

$$M_t = \mathbb{E}\big(X \,\big|\, \mathcal{F}_t\big)$$

is a Brownian martingale for every $X \in L_1(\mathcal{F}_\infty)$.

4b9 Exercise. Prove that the following process is a Brownian martingale:

$$M(t) = \begin{cases} \int_0^t B(s)\,\mathrm{d}s + (1-t)B(t) & \text{if } t \in [0,1], \\ \int_0^1 B(s)\,\mathrm{d}s & \text{if } t \in [1,\infty). \end{cases}$$

4b10 Theorem. If f satisfies the conditions of Lemma 4a5 then the following process is a Brownian martingale:

$$M(t) = f\big(t, B(t)\big) - \int_0^t g\big(s, B(s)\big)\,\mathrm{d}s\,,$$

where $g = f_{1,0} + \frac{1}{2} f_{0,2}$, that is,

$$g(t,x) = \Big(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big) f(t,x)\,.$$

[1] A Brownian martingale is a martingale w.r.t. the Brownian filtration $(\mathcal{F}_t)_t$. Generally, a martingale w.r.t. a given filtration is defined similarly.

4b11 Exercise. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\int_0^t g\big(s,B(s)\big)\,\mathrm{d}s = \mathbb{E}\,g\big(t,B(t)\big) \quad \text{for } t>0\,.$$

4b12 Exercise. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\,M(t) = 0 \quad \text{for } t>0\,.$$

4b13 Exercise. Prove Theorem 4b10.

4c Nothing happens suddenly to Brownian motion

4c1 Theorem. Every Brownian martingale can be upgraded[1],[2] to a random continuous function.

Similarly to the Brownian motion itself, we always upgrade $(M_t)_t$ this way.

Sample functions of a Brownian martingale are continuous.

4c2 Corollary. For every stopping time T such that $T>0$ a.s. there exist stopping times $T_1, T_2, \dots$ such that almost surely

$$T_n \uparrow T\,, \quad T_n < T\,.$$

Nothing happens suddenly to Brownian motion.

A striking contrast to jumping processes! I mean the Poisson process, the special Levy process, the Cauchy process.

Proof of Corollary 4c2. The process[3]

$$M_t = \mathbb{E}\big(\tfrac{T}{T+1}\,\big|\,\mathcal{F}_t\big)$$

is a Brownian martingale. By Theorem 4c1 it is continuous a.s. Clearly, $M_T = \frac{T}{T+1}$ and moreover,

$$T = \inf\{t : M_t - \tfrac{t}{t+1} = 0\}$$

[1]In the sense discussed in 1e (after 1e6).

[2]Generally, a martingale (w.r.t. any filtration) can be upgraded to a random r.c.l.l function.

[3]The idea is, to consider the expected remaining time, $\mathbb{E}\big(T-t\,\big|\,\mathcal{F}_t\big)$. However, T need not be integrable.

(think, why). We take

$$T_n = \min\{t \in [0,\infty) : M_t - \tfrac{t}{t+1} \le \tfrac1n\} \,.$$

$\square$

Now we have to prove Theorem 4c1.

We consider $(M_t)_{t\in[0,1]}$ (larger intervals are treated similarly); thus,

$$M(t) = \mathbb{E}\big(X \,\big|\, \mathcal{F}_t \big)$$

for some $X \in L_1(\mathcal{F}_1)$ (namely, $X = M_1$). Our first goal is to show that it is sufficient to ensure a.s. continuity of $M(\cdot)$ for a dense set of $X \in L_1(\mathcal{F}_1)$.

4c3 Proposition. For every $X \in L_1(\mathcal{F}_1)$ there exist $X_n \in L_1(\mathcal{F}_1)$ such that

(a) $\mathbb{E}\,|X_n - X| \to 0$ as $n \to \infty$,

(b) each martingale

$$M_n(t) = \mathbb{E}\big(X_n \,\big|\, \mathcal{F}_t \big)$$

can be upgraded to a random continuous function,

(c) $M_n(\cdot)$ converge in $C[0,1]$ almost surely; that is,

$$\max_{t\in[0,1]} |M_n(t) - M_\infty(t)| \to 0 \quad \text{as } n \to \infty$$

for some random continuous function $M_\infty(\cdot)$.

Proof of Th. 4c1, given Prop. 4c3. The equivalence class $M(t)$ contains $M_\infty(t)$, since

$$\mathbb{E}\,|M(t) - M_n(t)| = \mathbb{E}\,|\mathbb{E}\big(X - X_n \,\big|\, \mathcal{F}_t \big)| \le \\ \le \mathbb{E}\,\big(\mathbb{E}\big(|X - X_n| \,\big|\, \mathcal{F}_t \big) \big) = \mathbb{E}\,|X - X_n| \to 0$$

and therefore $M_{n_k}(t) \to M(t)$ a.s. (for an appropriate subsequence). $\square$

4c4 Exercise. Let $(M_t)_t$ be a Brownian martingale, and $T \subset [0,1]$ a countable set. Then

$$\mathbb{P}\Big(\sup_{t\in T} |M(t)| \ge c \Big) \le \frac{\mathbb{E}\,|M(1)|}{c} \quad \text{for } c \in (0,\infty)\,.$$

Prove it.

4c5 Exercise. Prove Item (c) of Prop. 4c3, assuming Items (a) and (b).

It remains to ensure a.s. continuity (in t) of $M(t) = \mathbb{E}\big(X \,\big|\, \mathcal{F}_t\big)$ for all X of a dense set of $\mathcal{X} \subset L_1(\mathcal{F}_1)$. Moreover, it is enough if linear combinations of these X are dense. There are several reasonable choices of such $\mathcal{X}$. You may try bounded uniformly continuous functions $C[0,\infty) \to \mathbb{R}$, or $X = f\big(B(t_1),\dots,B(t_n)\big)$ for bounded continuous $f : \mathbb{R}^n \to \mathbb{R}$, or $X = \int_0^1 \exp\big(\mathrm{i}f(t)B(t)\big)\,\mathrm{d}t$ etc. I prefer indicators $X = \mathbb{1}_A$ of events of the form

$$A = \{a_1 \le B(t_1) \le b_1, \dots, a_n \le B(t_n) \le b_n\}\,. \tag{4c6}$$

4c7 Exercise. Let $0 < t_1 < \dots < t_n < 1$, and $-\infty < a_k < b_k < \infty$ for $k = 1,\dots,n$. Then the random function

$$M(t) = \mathbb{E}\bigg(\prod_{k=1}^n \mathbb{1}_{[a_k,b_k]}\big(B(t_k)\big) \,\bigg|\, \mathcal{F}_t \bigg)$$

is continuous a.s.

Prove it.

Linear combinations of (arbitrary) indicators are dense in $L_1(\mathcal{F}_1)$. It remains to prove that sets of the form (4c6) and their disjoint unions are dense in $\mathcal{F}_1$ (mod 0) according to the metric $\operatorname{dist}(A,B) = \mathbb{P}\big(A \setminus B\big) + \mathbb{P}\big(B \setminus A\big)$. This claim follows easily from the next general lemma.

4c8 Lemma. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{A} \subset \mathcal{F}$ an algebra of sets that generates $\mathcal{F}$ (mod 0). Then:

(a) For every $C \in \mathcal{F}$ and $\varepsilon > 0$ there exist $A_1, A_2, \dots \in \mathcal{A}$ such that $C \subset A_1 \cup A_2 \cup \dots$ and $P(A_1 \cup A_2 \cup \dots) \le P(C) + \varepsilon$. In other words, there exists $A \in \mathcal{A}_\sigma$ such that $C \subset A$ and $P(A) \le P(C) + \varepsilon$.

(b) For every $C \in \mathcal{F}$ there exist $B_1, B_2, \dots \in \mathcal{A}_\sigma$ such that $B_1 \supset B_2 \supset \dots$, $C \subset B_1 \cap B_2 \dots$, and $P(C) = P(B_1 \cap B_2 \dots)$. In other words, there exists $B \in \mathcal{A}_{\sigma\delta}$ such that $C \subset B$ and $P(B \setminus C) = 0$.

(c) For every $C \in \mathcal{F}$ and $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that $P(A \setminus C) + P(C \setminus A) \le \varepsilon$.

See [1], Appendix A.3.

The proof of Theorem 4c1 is thus finished.

On the other hand, our approach to conditioning gives us another way of upgrading a martingale to a random function, at least when $X = M(1) = \varphi\big(B(\cdot)\big)$ for a bounded Borel function $\varphi : C[0,1] \to \mathbb{R}$.[1] Namely,

$$M(t)(\omega_1) = \mathbb{E}\big(X \,\big|\, \mathcal{F}_t\big)(\omega_1) = \int \varphi\Big(B(\cdot)(\omega_1) \overset{t}{\bigsqcup} B(\cdot)(\omega_2) \Big) P(\mathrm{d}\omega_2)\,. \tag{4c9}$$

[1] I often write $\varphi\big(B(\cdot)\big)$ instead of $\varphi\big(B(\cdot)|_{[0,1]}\big)$.

Several questions appear naturally. What happens if we change φ on a negligible (w.r.t. the Wiener measure) set? What happens if φ is not bounded? And, above all: is it a random *continuous* function?

4c10 Exercise. Let $\varphi, \varphi_1, \varphi_2, \dots : C[0,1] \to [-1,1]$ be Borel functions such that $\varphi_n(x) \uparrow \varphi(x)$ (as $n \to \infty$) for every $x \in C[0,1]$, and $M, M_1, M_2, \dots$ random functions corresponding to $\varphi, \varphi_1, \varphi_2, \dots$ according to (4c9). If each M_k is a.s. continuous, then M is a.s. continuous.

Prove it.

Due to 4c7 we have an algebra $\mathcal{A} \subset \mathcal{B}_1$ such that

(a) $\mathcal{A}$ generates $\mathcal{B}_1$,

(b) for every $A \in \mathcal{A}$ the random function $M_A(t) = \mathbb{P}\big(A \big| \mathcal{F}_t\big)$ is a.s. continuous.

By 4c10, $M_B(\cdot)$ is a.s. continuous for all $B \in \mathcal{A}_\sigma$ and moreover, all $B \in \mathcal{A}_{\sigma\delta}$.

4c11 Exercise. If $C \in \mathcal{B}_1$ is negligible in the sense that $\mathbb{P}\big(B(\cdot) \in C\big) = 0$, then $\mathbb{P}\big(\forall t \; M_C(t) = 0\big) = 1$.

Prove it.

Combining 4c11 and 4c8(b) we get the following.

4c12 Theorem. For every Borel set $G \subset C[0,1]$, the following random function is a.s. continuous:

$$M_G(t) = \mathbb{P}\big(B(\cdot) \in G \big| \mathcal{F}_t\big) \, ,$$

that is,

$$M_G(t)(\omega_1) = \mathbb{P}\Big(\Big\{\omega_2 : B(\cdot)(\omega_1) \bigsqcup^t B(\cdot)(\omega_2) \in G\Big)\Big\} \, .$$

4c13 Exercise. $\forall G \quad \mathbb{P}\big(M_G(\cdot) \text{ is continuous }\big) = 1$, however, $\mathbb{P}\big(\forall G \;\; M_G(\cdot) \text{ is continuous }\big) = 0$; here G runs over all Borel subsets of $C[0,1]$.

Prove it.

4c14 Exercise. Let $A \in \mathcal{F}_t$ for all $t > 0$; then either $\mathbb{P}\big(A\big) = 0$ or $\mathbb{P}\big(A\big) = 1$. ('Blumenthal's $0-1$ law')

Prove it.

In other words, $\mathcal{F}_{0+} = \mathcal{F}_0 \pmod 0$. See also [1], Sect. 7.2, (2.5).

4c15 Exercise. Give another proof to 2b6, using 4c14 (and not the distribution of T_x).

4c16 Exercise. For every $f : (0,\infty) \to (0,\infty)$ the random variable

$$\limsup_{t\to 0+} \frac{B(t)}{f(t)} \in [0,\infty]$$

is degenerate (that is, equal a.s. to a constant).

Prove it.

4c17 Exercise. Generalize 4c14 for $A \in \cap_{\varepsilon>0} \mathcal{F}_{T+\varepsilon}$, where T is a stopping time. That is, show that $\mathcal{F}_{T+} = \mathcal{F}_T \pmod 0$.

4c18 Exercise. Generalize 4c10 to the case when $\varphi_k : C[0,1] \to [0,\infty)$ are bounded, while $\varphi : C[0,1] \to [0,\infty)$ need not be bounded, but $\mathbb{E}\,\varphi\big(B(\cdot)\big) < \infty$.

4c19 Exercise. Let $\varphi : C[0,1] \to \mathbb{R}$ be a Borel function such that $\mathbb{E}\,\varphi\big(B(\cdot)\big) < \infty$. Then the random function

$$M(t) = \mathbb{E}\left(\varphi\big(B(\cdot)\big)\,\middle|\,\mathcal{F}_t\right)$$

is a.s. continuous.

Prove it.

4d Hints to exercises

4a12: $\mathbb{E}\,\big(B^4(T\wedge n) - 6(T\wedge n)B^2(T\wedge n)\big) = -3(T\wedge n)^2$; use T as an integrable majorant in the left-hand side...

4a13: $\mathbb{E}\,\big(\mathrm{e}^{\lambda^2 T/2} 1\!\!1_{[0,t]}(T)\big) \le \frac{1}{\cos\lambda}$ for all $\lambda \in [0,\pi/2)$ and $t \in (0,\infty)$.

4c4: recall 3c11.

4c5: $\sum_n \mathbb{P}\,\big(\max_{[0,1]} |M_n - M_{n+1}| \ge 1/n^2\big) \le \sum_n n^2 \mathbb{E}\,|X_n - X_{n+1}| < \infty$.

4c7: Consider the integral of the density; note that $B(t_k) \ne a_k$, $B(t_k) \ne b_k$ a.s.

4c10: By (4c9) and the bounded convergence theorem, $\mathbb{P}\big(\forall t \in [0,1]\ M_n(t) \uparrow M(t)\big) = 1$. On the other hand, similarly to 4c5, $M_n(\cdot)$ converge in $C[0,1]$ to some $M_\infty(\cdot)$ a.s., if $\varphi_n \to \varphi$ fast enough (take a subsequence). Therefore $M(\cdot) = M_\infty(\cdot)$ a.s.

4c11: 4c8(b) gives a negligible $B \in \mathcal{A}_{\sigma\delta}$ such that $C \subset B$; thas, $M_C(\cdot) \le M_B(\cdot)$, $M_B(\cdot)$ is continuous, and $\mathbb{E}\,M_B(t) = \mathbb{E}\,M_B(1) = 0$.

4c13: Consider $G = G_x = \{f : f(1) \le x\}$ for all $x \in \mathbb{R}$.

4c14: consider $\mathbb{P}\big(A\,\big|\,\mathcal{F}_t\big)$.

4c19: $\varphi = \varphi^+ - \varphi^-$; use 4c18, each φ_k being a linear combination of indicators.

References

[1] R. Durrett, *Probability: theory and examples,* 1996.

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