5 Localization

Local aspects (derivatives) and global aspects (growth on infinity) are entangled in Sect. 4. Now we'll see how to disentangle them.

5a Heat equation localized

5a1 Lemma. Let $K \subset [0,\infty) \times \mathbb{R}$ be a compact set, and T a stopping time such that[1]

$$\mathbb{P}\big(\forall t \;\; (t \wedge T, B(t \wedge T)) \in K\big) = 1 \,.$$

Let $G \subset [0,\infty) \times \mathbb{R}$ be a relatively open set,[2] $G \supset K$, and $u : G \to \mathbb{R}$ a continuous function having continuous derivatives $u_{1,0}, u_{0,1}, u_{0,2}$ and satisfying the PDE $u_{1,0} + \frac{1}{2} u_{0,2} = 0$.[3] Then the following process is a martingale:

$$M(t) = u\big(t \wedge T, B(t \wedge T)\big) \,.$$

The proof will be given after some preparation. If T is a stopping time such that $\exists t \; \mathbb{P}\big(T \le t\big) = 1$, then (recall 3a5 and 4a10)

$$\mathbb{E}\, B(T) = 0 \,, \quad \mathbb{E}\, B^2(T) = \mathbb{E}\, T \,. \tag{5a2}$$

5a3 Exercise. Let T be a stopping time, $f \in C[0,\infty)$, and $t \in [0, T(f)]$. Then the function

$$g \mapsto T(f \overset{t}{\sqcup} g) - t$$

is a stopping time.

Prove it.

[1] It is evidently equivalent to $\forall t \;\; \mathbb{P}\big((t \wedge T, B(t \wedge T)) \in K\big) = 1$. As before, $t \wedge T$ means $\min(t, T)$.

[2] Just the intersection of the closed half-plane $[0,\infty) \times \mathbb{R}$ and an open subset of the plane $\mathbb{R}^2$.

[3] As before, $f_{i,j}(t,x) = \frac{\partial^{i+j}}{\partial t^i \partial x^j} f(t,x)$.

5a4 Exercise. Let T_1, T_2 be stopping times, $\exists t \; \mathbb{P}\big(T_2 \le t\big) = 1$. Then the equalities

$$\mathbb{E}\big(B(T_2) - B(T_1)\big|\mathcal{F}_{T_1}\big) = 0\,, \tag{5a5}$$

$$\mathbb{E}\big((B(T_2) - B(T_1))^2\big|\mathcal{F}_{T_1}\big) = \mathbb{E}\big(T_2 - T_1\big|\mathcal{F}_{T_1}\big) \tag{5a6}$$

hold almost surely on the event $\{T_1 \le T_2\}$.

Prove it.

Proof of Lemma 5a1. Denote $M(t) = u(t \wedge T, B(t \wedge T))$. We have to prove that $\mathbb{E}\big(M(t)\big|\mathcal{F}_s\big) = M(s)$ for $s \le t$. It is sufficient to prove that

$$\mathbb{E}\big(M(t)\big|\mathcal{F}_s\big) - M(s) = o(t-s) \quad \text{a.s. for } s \le t\,; \tag{5a7}$$

here and henceforth all $o(\dots)$ are uniform (in everything; this time, in s, t, ω). Here is why (5a7) is sufficient:

$$\mathbb{E}\big(\mathbb{E}\big(M(t+\varepsilon)\big|\mathcal{F}_t\big)\big|\mathcal{F}_s\big) = \mathbb{E}\big(M(t+\varepsilon)\big|\mathcal{F}_s\big)$$

(think, why), thus,

$$\begin{aligned}|\mathbb{E}\big(M(t+\varepsilon)\big|\mathcal{F}_s\big) - \mathbb{E}\big(M(t)\big|\mathcal{F}_s\big)| = |\mathbb{E}\big(\mathbb{E}\big(M(t+\varepsilon)\big|\mathcal{F}_t\big) - M(t)\big|\mathcal{F}_s\big)| \le \\ \le \mathbb{E}\big(|\mathbb{E}\big(M(t+\varepsilon)\big|\mathcal{F}_t\big) - M(t)|\big|\mathcal{F}_s\big) = \mathbb{E}\big(o(\varepsilon)\big|\mathcal{F}_s\big) = o(\varepsilon)\,.\end{aligned}$$

It remains to prove (5a7).

On the event $\{T < s\}$ we have

$$\mathbb{E}\big(M(t)\big|\mathcal{F}_s\big) - M(s) = \mathbb{E}\big(u(T, B(T))\big|\mathcal{F}_s\big) - u(T, B(T)) = 0 \quad \text{a.s.},$$

thus, it is sufficient to prove (5a7) on the event $\{T \ge s\}$. From now on we assume $T \ge s$.

We define R by

$$\begin{aligned}u(t \wedge T, B(t \wedge T)) - u(s, B(s)) = u_{1,0}(s, B(s))(t \wedge T - s) + \\ + u_{0,1}(s, B(s))(B(t \wedge T) - B(s)) + \frac{1}{2} u_{0,2}(s, B(s))(B(t \wedge T) - B(s))^2 + R\,,\end{aligned}$$

take $\mathbb{E}\big(\dots\big|\mathcal{F}_s\big)$, use (5a5), (5a6) and get

$$\begin{aligned}\mathbb{E}\big(M(t)\big|\mathcal{F}_s\big) - M(s) = u_{1,0}(s, B(s))\mathbb{E}\big(t \wedge T - s\big|\mathcal{F}_s\big) + \\ + u_{0,1}(s, B(s)) \cdot 0 + \frac{1}{2} u_{0,2}(s, B(s)) \cdot \mathbb{E}\big(t \wedge T - s\big|\mathcal{F}_s\big) + \mathbb{E}\big(R\big|\mathcal{F}_s\big) = \\ = \Big(u_{1,0} + \frac{1}{2} u_{0,2}\Big)(s, B(s)) \cdot \mathbb{E}\big(t \wedge T - s\big|\mathcal{F}_s\big) + \mathbb{E}\big(R\big|\mathcal{F}_s\big) = \mathbb{E}\big(R\big|\mathcal{F}_s\big)\,;\end{aligned}$$

it remains to check that $\mathbb{E}\big(R\big|\mathcal{F}_s\big) = o(t-s)$.

We have

$$R = o(t\wedge T - s) + o\big((B(t\wedge T) - B(s))^2\big)\,;$$

$o(\dots)$ are uniform (in s,t,ω) since $u_{1,0}$ and $u_{0,2}$ are uniformly continuous on K. Clearly, $t\wedge T - s \le t - s$. It remains to prove that

$$\int_{-\infty}^{+\infty} \big(o(x^2)\wedge C\big) p_\varepsilon(x)\,\mathrm{d}x = o(\varepsilon)\,.$$

The integral over $\mathbb{R}\setminus[-\delta,\delta]$ is exponentially small. The integral over $[-\delta,\delta]$ is much smaller than $\int x^2\,p_\varepsilon(x)\,\mathrm{d}x = \varepsilon$ if δ is small enough. □

We may generalize 5a1 in the spirit of 4b10.

5a8 Lemma. Let $K \subset [0,\infty)\times\mathbb{R}$ be a compact set, and T a stopping time such that

$$\mathbb{P}\big(\forall t\ \ (t\wedge T, B(t\wedge T)) \in K\big) = 1\,.$$

Let $G\subset[0,\infty)\times\mathbb{R}$ be a relatively open set, $G\supset K$, and $u : G\to\mathbb{R}$ a continuous function having continuous derivatives $u_{1,0}, u_{0,1}, u_{0,2}$. Then the following process is a martingale:

$$M(t) = u\big(t\wedge T, B(t\wedge T)\big) - \int_0^{t\wedge T} v(s,B(s))\,\mathrm{d}s\,,$$

where $v = u_{1,0} + \frac12 u_{0,2}$.

Proof. Similarly to the proof of 5a1 we get

$$\begin{aligned}
&\mathbb{E}\big(M(t)\big|\mathcal{F}_s\big) - M(s) = \\
&= v(s,B(s))\cdot\mathbb{E}\big(t\wedge T - s\big|\mathcal{F}_s\big) + \mathbb{E}\big(R\big|\mathcal{F}_s\big) - \mathbb{E}\bigg(\int_s^{t\wedge T} v(r,B(r))\,\mathrm{d}r\bigg|\mathcal{F}_s\bigg) = \\
&\qquad = \mathbb{E}\big(R\big|\mathcal{F}_s\big) - \mathbb{E}\bigg(\int_s^{t\wedge T} \big(v(r,B(r)) - v(s,B(s))\big)\,\mathrm{d}r\bigg|\mathcal{F}_s\bigg)\,.
\end{aligned}$$

By the uniform continuity of v on K, for every ε there exists δ such that $|v(r,B(r)) - v(s,B(s))| \le \varepsilon$ whenever $|r-s|\le\delta$ and $|B(r)-B(s)|\le\delta$. Assuming $t-s\le\delta$ we have

$$\begin{aligned}
&\mathbb{E}\bigg(\int_s^{t\wedge T} |v(r,B(r)) - v(s,B(s))|\,\mathrm{d}r\bigg|\mathcal{F}_s\bigg) \le \\
&\qquad \le \varepsilon(t\wedge T - s) + 2\Big(\max_K|v(\cdot)|\Big)\mathbb{P}\bigg(\max_{[s,t\wedge T]}|B(\cdot) - B(s)| > \delta\bigg|\mathcal{F}_s\bigg) \le \\
&\qquad\qquad \le \varepsilon(t-s) + o(t-s)\,.
\end{aligned}$$

Therefore it is $o(t-s)$. □

5b Local martingales

5b1 Definition. A *Brownian local martingale*[1] is a random continuous function $(M_t)_{t\in[0,\infty)}$ (on a probability space carrying a Brownian motion $(B_t)_t$) such that there exists a sequence of stopping times $T_1, T_2, \dots$ (so-called localizing sequence) satisfying

$$T_n \uparrow +\infty \quad \text{a.s.};$$
$$(M_{t\wedge T_n})_t \quad \text{is a Brownian martingale (for each } n\text{)}.$$

5b2 Proposition. Let $u : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a continuous function having continuous derivatives $u_{1,0}, u_{0,1}, u_{0,2}$. Then the following process is a Brownian local martingale:

$$M(t) = u\big(t, B(t)\big) - \int_0^t v\big(s, B(s)\big)\, \mathrm{d}s\,,$$

where $v = u_{1,0} + \frac12 u_{0,2}$.

Proof. Let $T_n = \inf\{t : (t, B(t)) \notin [0,n)\times(-n,n)\}$, then clearly $T_n \uparrow \infty$, and $\big(M(t\wedge T_n)\big)_t$ is a martingale by Lemma 5a8. □

5b3 Corollary. Let u satisfy the conditions of Prop. 5b2 and the PDE $u_{1,0} + \frac12 u_{0,2} = 0$. Then the process $M(t) = u(t, B(t))$ is a local martingale.

Recall Tychonoff's counterexample mentioned in 4a (after 4a12); it is a function that satisfies the PDE (4a4) but violates (4a1). By 5b3 it leads to a local martingale that is not a martingale. Somehow, the expectation escapes to the spatial infinity when $t \to 1-$.[2]

5b4 Exercise. The following is a local martingale but not a martingale:[3]

$$M(t) = \begin{cases} p_{1-t}(B(t)) & \text{for } t\in[0,1),\\ 0 & \text{for } t\in[1,\infty). \end{cases}$$

Prove it.

---

[1]This is a local martingale w.r.t. the Brownian filtration $(\mathcal{F}_t)_t$. Generally, a local martingale w.r.t. a given filtration is defined similarly, but need not be continuous (rather, r.c.l.l.). I often omit the word 'Brownian'.

[2]In reversed time, heat comes from the spatial infinity by a giant fast oscillating heat wave. A terrible spectacle!

[3]As before, $p_t(x) = (2\pi t)^{-1/2}\exp(-\frac{x^2}{2t})$.

5b5 Proposition. Let $(M_t)_t$ be a local martingale, $(T_n)_n$ a localizing sequence, and

$$\sup_n \mathbb{E}\, M^2_{t\wedge T_n} < \infty \quad \text{for all } t\,.$$

Then $(M_t)_t$ is a martingale.

5b6 Corollary. A local martingale $(M_t)_t$ satisfying

$$\mathbb{E} \max_{s\in[0,t]} M_s^2 < \infty \quad \text{for all } t$$

is a martingale.

5b7 Exercise. Prove that

$$\|M_{t\wedge T_{n+k}} - M_{t\wedge T_n}\|_1 \le 2\sqrt{\mathbb{P}\big(T_n < t\big)}\big(\|M_{t\wedge T_{n+k}}\|_2 + \|M_{t\wedge T_n}\|_2\big)\,.$$

5b8 Exercise. Prove that $M_t \in L_1$ and $M_{t\wedge T_n} \to M_t$ in L_1 as $n\to\infty$.

5b9 Exercise. Prove Prop. 5b5.

The condition $\mathbb{E}\, M_t^2 < \infty$ on a local martingale does not guarantee that it is a martingale! This condition fails for 5a10 (and Tychonoff's counterexample), however, later (in Sect. 6c) we'll see a local martingale $M(\cdot)$ satisfying $\sup_{t\in[0,\infty)} \mathbb{E}\, \mathrm{e}^{|M(t)|} < \infty$ but still not a martingale.[1]

5c Heat equation revisited

5c1 Theorem. [2] Let u satisfy the conditions of Prop. 5b2. Assume that

$$\frac{1}{x^2}\ln^+|u(t,x)| \to 0 \quad \text{as } x\to\pm\infty\,,$$
$$\frac{1}{x^2}\ln^+|v(t,x)| \to 0 \quad \text{as } x\to\pm\infty$$

uniformly in $t\in[0,b]$ for every b; here $v = u_{1,0} + \frac12 u_{0,2}$, that is,

$$v(t,x) = \Big(\frac{\partial}{\partial t} + \frac12\frac{\partial^2}{\partial x^2}\Big)u(t,x)\,.$$

Then the following process is a Brownian martingale:

$$M(t) = u\big(t,B(t)\big) - \int_0^t v\big(s,B(s)\big)\,\mathrm{d}s\,.$$

[1] "…we stress the fact that local martingales are *much more general* than martingales and warn the reader against the common mistaken belief that local martingales need only be integrable in order to be martingales." [1] page 117.

[2] See also [2], p. 36.

Theorem 4b10 is thus generalized; the condition $\frac{1}{x^2}\ln^+|u_{i,j}(t,x)| \to 0$ appears to be unnecessary (unless $(i,j) = (0,0)$).

Note especially the case $v = 0$.

Also Prop. 4a9 is now generalized: Condition (4a1) is equivalent to the PDE (4a4) for all functions satisfying the conditions of Theorem 5c1.

Proof of Theorem 5c1. Let $T_n = \inf\{t : (t, B(t)) \notin [0,n) \times (-n,n)\}$ (as in the proof of 5b2). We have

$$\mathbb{P}\Big(\max_{[0,t]} |B(\cdot)| \ge c \Big) \le 2\,\mathbb{P}\Big(\max_{[0,t]} B(\cdot) \ge c \Big) = 2\,\mathbb{P}\big(|B(t)| \ge c \big)\,;$$
$$|u\big(t, B(t)\big)| \le C_\delta \exp\big(\delta B^2(t)\big)\,, \quad |v\big(t, B(t)\big)| \le C_\delta \exp\big(\delta B^2(t)\big)$$

and we may choose $\delta > 0$ at will. Thus,

$$\mathbb{E} \max_{[0,t]} M^2(\cdot) \le \mathbb{E} \bigg(C_\delta \exp\Big(\delta \max_{[0,t]} B^2(\cdot)\Big) + tC_\delta \exp\Big(\delta \max_{[0,t]} B^2(\cdot)\Big)\bigg)^2 =$$
$$= C_\delta^2 (1+t)^2 \mathbb{E}\,\exp 2\delta \max_{[0,t]} B^2(\cdot) \le C_\delta^2 (1+t)^2 \cdot 2\,\mathbb{E}\,\exp 2\delta B^2(t) < \infty$$

if δ is small enough (namely, $2\delta < \frac{1}{2t}$). Cor. 5b6 completes the proof. □

5d Finite lifetime

5d1 Definition. Let T be a stopping time. A *random continuous function on* $[0,T)$ is a function[1]

$$X : \{(t,\omega) \in [0,\infty) \times \Omega : t < T(\omega)\} \to \mathbb{R}$$

such that for every t the function $X(t,\cdot)$ on $\{\omega : T(\omega) > t\}$ is measurable, and for almost every ω the function $X(\cdot,\omega)$ on $\big[0, T(\omega)\big)$ is continuous.

5d2 Definition. A random continuous function on $[0,T)$ is a *Brownian local martingale*[2] *on* $[0,T)$ if there exists a sequence of stopping times $T_1, T_2, \dots$ (called localizing sequence) satisfying

$$T_n < T \quad \text{and} \quad T_n \uparrow T \quad \text{a.s.;}$$
$$(M_{t \wedge T_n})_t \quad \text{is a Brownian martingale (for each } n).$$

[1] Or rather, equivalence class.

[2] A martingale, in contrast to a local martingale, is defined on the whole $[0,\infty)$.

5d3 Proposition. Let $G \subset [0,\infty)\times\mathbb{R}$ be a relatively open set, T a stopping time,

$$\mathbb{P}\big(\forall t \in [0,T) \;\; (t,B(t)) \in G\big) = 1\,,$$

$u : G \to \mathbb{R}$ a continuous function having continuous derivatives $u_{1,0}, u_{0,1}, u_{0,2}$. Then the following process is a Brownian local martingale on $[0,T)$:

$$M(t) = u\big(t,B(t)\big) - \int_0^t v\big(s,B(s)\big)\,\mathrm{d}s \quad \text{for } t \in [0,T)\,,$$

where $v = u_{1,0} + \frac{1}{2}u_{0,2}$.

Proof. We take relatively open sets $G_1 \subset G_2 \subset \dots \subset G$ such that $(0,0) \in G_1$, $G_1 \cup G_2 \cup \dots = G$ and the closure $\overline{G}_n$ of G_n is a compact subset of G (for each n).[1] We define stopping times $T_n = \inf\{t : t \ge T \text{ or } (t,B(t)) \notin G_n\}$ and observe that $T_n \uparrow T$ a.s. (since otherwise a compact curve is included in G but not in any G_n). By Lemma 5a8 (applied to $\overline{G}_n$ and T_n) the process $t \mapsto M(t \wedge T_n)$ is a martingale. □

5e Hints to exercises

5a3: recall Def. 2f5.

5a4: use 5a3.

5b4: The closed set $\{(t,B(t)) : t \in [0,\infty)\}$ a.s. does not contain $(1,0)$.

5b7: $\|1\!\!1_{T_n<t}\|_2 = \sqrt{\mathbb{P}\big(T_n < t\big)}$.

5b8: $M_{t\wedge T_n}$ converges to something in L_1, and to M_t a.s.

5b9: $\mathbb{E}\big(M_{t\wedge T_n}\big|\mathcal{F}_s\big) \to \mathbb{E}\big(M_t\big|\mathcal{F}_s\big)$ in L_1.

References

[1] D. Revuz, M. Yor, *Continuous martingales and Brownian motion* (second edition), 1994.

[2] L.C.G. Rogers, D. Williams, *Diffusions, Markov processes, and martingales,* vol. 1 (second edition), 1994.

Index

[1]For example: G_n consists of all points $(t,x) \in G$ such that $t < n$, $|x| < n$, and the closed $1/n$-neighborhood of (t,x) in $[0,\infty)\times\mathbb{R}$ is contained in G.