2 Random real zeroes

2a No derivatives

We consider random trigonometric sums of the form

$$X(t) = \sum_{k=1}^{N} a_k \operatorname{Re}\big((X_{2k-1} + \mathrm{i} X_{2k}) \mathrm{e}^{\mathrm{i}\lambda_k t} \big) = \sum_{k=1}^{N} a_k (X_{2k-1} \cos \lambda_k t - X_{2k} \sin \lambda_k t)$$

where $a_1, \dots, a_N > 0$, $0 < \lambda_1 < \dots < \lambda_N < \infty$, and $X_1, \dots, X_{2N}$ are independent standard normal (that is, distributed $N(0,1)$) random variables.

The distribution of $X(\cdot)$ is shift-invariant. In other words, $X(\cdot)$ is a stationary random process.

It may happen that all $\lambda_k/(2\pi)$ are integers, and then $X(t+1) = X(t)$, but generally $X(\cdot)$ need not be periodic.

Assumption A:

$$\sum_{k=1}^{N} a_k^2 = 1 \, .$$

That is, $X(0) \sim N(0,1)$. Otherwise we may rescale X.

Assumption A_n: assumption A holds, and in addition,

$$\forall \lambda \in [0,\infty) \qquad \sum_{k:\lambda_k \in [\lambda, \lambda+1]} a_k^2 \le \frac{1}{n} \, .$$

The correlation function

$$\mathbb{E}\big(X(0) X(t) \big) = \sum_{k=1}^{N} a_k^2 \cos \lambda_k t$$

[1]This section is, to a large extent, a one-dimensional counterpart of the work: F. Nazarov, M. Sodin (2009), "On the number of nodal domains of random spherical harmonics", American Journal of Mathematics **131**:5, 1337–1357.

need not decay in n exponentially. For example, $N \gg n$, $\lambda_k = nk/N$, $a_k = 1/\sqrt{N}$; then

$$\mathbb{E}\,\bigl(X(0)X(t)\bigr) \approx \frac{1}{n}\int_0^n \cos \lambda t \, \mathrm{d}\lambda = \frac{\sin nt}{nt}\,;$$

another example gives

$$\mathbb{E}\,\bigl(X(0)X(t)\bigr) \approx \frac{2}{n^2}\int_0^n (n-\lambda)\cos \lambda t \, \mathrm{d}\lambda = \Bigl(\frac{\sin nt/2}{nt/2}\Bigr)^2 .$$

In amazing contrast, many interesting probabilities decay in n exponentially.

2a1 Lemma. Let X satisfy assumption A, and a measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ be γ^1-integrable (that is, $\int |\varphi| \, \mathrm{d}\gamma^1 < \infty$). Then the random variable

$$\xi = \int_0^1 \varphi\bigl(X(t)\bigr) \, \mathrm{d}t$$

is integrable, and

$$\mathbb{E}\,\xi = \int \varphi \, \mathrm{d}\gamma^1 .$$

2a2 Theorem. Let X satisfy assumption A_n, and a function $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere, and

$$\sup_x \frac{|\varphi(x)|}{1+|x|} < \infty .$$

Then the random variable ξ introduced above satisfies, for every $\varepsilon > 0$,

$$\mathbb{P}\bigl(|\xi - \mathbb{E}\,\xi| \ge \varepsilon\bigr) \le 2\mathrm{e}^{-c_{\varepsilon,\varphi} n}$$

for some $c_{\varepsilon,\varphi} > 0$ (dependent on ε and φ only, not on n).

The same holds for $\mathbb{R}^d$-valued processes $X(\cdot)$ provided that $X(0) \sim \gamma^d$.

For $f \in C[0,1]$ denote

$$T(f) = \inf_g \int_0^1 |f(t) - g(t)| \, \mathrm{d}t$$

where the infimum is taken over all measurable $g : (0,1) \to \mathbb{R}$ that send Lebesgue measure to γ^1.

2a3 Theorem. Let X satisfy assumption A_n. Then

$$\mathbb{P}\bigl(T(X(\cdot)) \ge \varepsilon\bigr) \le 2\mathrm{e}^{-c_\varepsilon n}$$

for some $c_\varepsilon > 0$ dependent on ε only.

The same holds for $\mathbb{R}^d$-valued processes $X(\cdot)$ (with γ^d in place of γ).

A trivial rescaling of t by arbitrary $L > 0$ turns Assumption A_n, Lemma 2a1 and Theorem 2a2 into the following.

Assumption $A_{n,L}$: assumption A holds, and in addition,

$$\forall \lambda \in [0,\infty) \qquad \sum_{k:\lambda_k \in [\lambda, \lambda + \frac{1}{L}]} a_k^2 \le \frac{1}{n} \,.$$

2a4 Lemma. Let X satisfy assumption A, and a measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ be γ^1-integrable. Then the random variable

$$\xi = \frac{1}{L} \int_0^L \varphi\big(X(t)\big) \, \mathrm{d}t$$

is integrable, and

$$\mathbb{E}\, \xi = \int \varphi \, \mathrm{d}\gamma^1 \,.$$

2a5 Theorem. Let X satisfy assumption $A_{n,L}$, and a function $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere, and

$$\sup_x \frac{|\varphi(x)|}{1+|x|} < \infty \,.$$

Then the random variable ξ introduced above satisfies, for every $\varepsilon > 0$,

$$\mathbb{P}\big(|\xi - \mathbb{E}\, \xi| \ge \varepsilon \big) \le 2\mathrm{e}^{-c_{\varepsilon,\varphi} n}$$

for some $c_{\varepsilon,\varphi} > 0$.

2b One derivative

Assumption B:

$$\sum_{k=1}^N a_k^2 = 1 \quad \text{and} \quad \sum_{k=1}^N \lambda_k^2 a_k^2 = 1 \,.$$

That is, $X(0) \sim N(0,1)$ and $X'(0) \sim N(0,1)$. Otherwise we may rescale t. In fact, $X(0)$ and $X'(0)$ are independent; thus, $\big(X(0), X'(0)\big) \sim \gamma^2$.

Assumption $B_{n,L}$: assumption B holds, and in addition,

$$\forall \lambda \in [0,\infty) \qquad \sum_{k:\lambda_k \in [\lambda, \lambda + \frac{1}{L}]} (1+\lambda_k^2) a_k^2 \le \frac{1}{n} \,.$$

If X satisfies $B_{n,L}$ then the 2-dimensional process (X, X') satisfies $A_{n,L}$.Thus, Theorem 2a2 (2-dim version) may be applied to random variables of the form

$$\frac{1}{L}\int_0^L \varphi\big(X(t), X'(t)\big)\, \mathrm{d}t\,.$$

But now we turn to a more interesting random variable.

2b1 Theorem. Let X satisfy assumption B, and a measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy $\int |\varphi(y)||y|\mathrm{e}^{-y^2/2}\, \mathrm{d}y < \infty$. Then the random variable

$$\xi = \frac{1}{L} \sum_{t\in[0,L], X(t)=0} \varphi\big(X'(t)\big)$$

is integrable, and

$$\mathbb{E}\,\xi = \frac{1}{2\pi}\int \varphi(y)|y|\mathrm{e}^{-y^2/2}\, \mathrm{d}y\,.$$

(Assumption B does not contain L, but anyway, the theorem above holds for all $L > 0$.)

In particular (for $\varphi(\cdot) = 1$), the expected number of zeroes per unit time is equal to $1/\pi$.[1]

The expected number (per unit time) of zeroes t such that $|X'(t)| \le \varepsilon$ is $O(\varepsilon^2)$ as $\varepsilon \to 0+$.

2c Two derivatives

Assumption C_M:

$$\sum_{k=1}^N a_k^2 = 1, \quad \sum_{k=1}^N \lambda_k^2 a_k^2 = 1\,, \quad \text{and} \quad \sum_{k=1}^N \lambda_k^4 a_k^2 \le M\,.$$

Thus, $\big(X(0), X'(0)\big) \sim \gamma^2$, and $\mathbb{E}\,|X''(0)|^2 \le M$. In fact, $X'(0)$ and $X''(0)$ are independent, but $\mathbb{E}\,\big(X(0)X''(0)\big) = -1$.

Assumption $C_{M,n,L}$: assumption C_M holds, and in addition,

$$\forall \lambda \in [0,\infty) \qquad \sum_{k:\lambda_k\in[\lambda,\lambda+\frac{1}{L}]} (1+\lambda_k^2)^2 a_k^2 \le \frac{1}{n}\,.$$

Clearly, $C_{M,n,L}$ implies $B_{n,L}$.

[1] "Rice's formula" (Kac 1943, Rice 1945, Bunimovich 1951, Grenander and Rosenblatt 1957, Ivanov 1960, Bulinskaya 1961, Itô 1964, Ylvisaker 1965 et al. See [1, Sect. 10.3]). "...the famous Rice formula, undoubtedly one of the most important results in the applications of smooth stochastic processes" (R.J. Adler and J.E. Taylor, "Random fields and geometry", Springer 2007; see Preface, page viii).

2c1 Theorem. Let X satisfy assumption $C_{M,n,L}$, and a function $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere, $\varphi(0) = 0$, and

$$\sup_{x \ne 0} \frac{|\varphi(x)|}{|x|} < \infty \,.$$

Then the random variable

$$\xi = \frac{1}{L} \sum_{t \in [0,L], X(t)=0} \varphi\big(X'(t)\big)$$

is integrable,

$$\mathbb{E}\,\xi = \frac{1}{2\pi} \int \varphi(y)|y|\mathrm{e}^{-y^2/2}\,\mathrm{d}y \,,$$

and

$$\mathbb{P}\big(|\xi - \mathbb{E}\,\xi| \ge \varepsilon\big) \le 2\mathrm{e}^{-c_{M,\varepsilon,\varphi} n}$$

for some $c_{M,\varepsilon,\varphi} > 0$.

References

[1] H. Cramér, M.R. Leadbetter, *Stationary and related stochastic processes,* Wiley 1967.