21 Gaussian spaces and processes

21a Gaussian spaces: finite dimension

21a1 Definition. [1] A (closed linear) subspace G of the (real) Hilbert space $L_2(\Omega, \mathcal{F}, P)$ over a probability space $(\Omega, \mathcal{F}, P)$ is called a *Gaussian Hilbert space* if each $g \in G$ satisfying $\|g\| = 1$ is distributed γ^1 (in other words, is a standard normal random variable).

I often abbreviate it to just "Gaussian space".
Clearly, a subspace of a Gaussian space is again a Gaussian space.

21a2 Example. Let $G = (\mathbb{R}^d)^* \subset L_2(\mathbb{R}^d, \gamma^d)$ be the d-dimensional space of all linear functions[2] $\mathbb{R}^d \to \mathbb{R}$. Then G is a Gaussian space. Each point $\omega \in \Omega = \mathbb{R}^d$ may be thought of as a linear function on G.

Let μ, ν be two probability measures on $\mathbb{R}^d$ such that $f[\mu] = f[\nu]$ for every linear function $f : \mathbb{R}^d \to \mathbb{R}$. Then $\mu = \nu$. (Fourier transform… "Characteristic function"…) Use this fact in the following exercise.

21a3 Exercise. Let G be a Gaussian space and $g_1, \dots, g_n \in G$ orthonormal vectors. Then $g_1, \dots, g_n$ are (not only standard normal but also) *independent* random variables; in other words, the map[3]

$$(\Omega, P) \to (\mathbb{R}^n, \gamma^n), \quad \omega \mapsto \big(g_1(\omega), \dots, g_n(\omega)\big)$$

is measure preserving.
Prove it.

[1]Not a widespread definition, but see the book "Gaussian Hilbert spaces" by Svante Janson.
[2]More exactly: their equivalence classes.
[3]More exactly: equivalence class of maps.

Every measure preserving map[1] $\alpha : (\Omega_1, P_1) \to (\Omega_2, P_2)$ leads to a linear isometric embedding $\tilde\alpha : L_2(\Omega_2, P_2) \to L_2(\Omega_1, P_1)$. But do not think that $\tilde\alpha$ is just an arbitrary linear isometry. No, it is very special: it preserves distributions! (Also joint distributions.) Therefore $\tilde\alpha$ sends every Gaussian space $G_2 \subset L_2(\Omega_2, P_2)$ into a Gaussian space $G_1 \subset L_2(\Omega_1, P_1)$; in this sense, G_1 is a copy of G_2.

By 21a3, every d-dimensional Gaussian space is a copy of the canonical one, $(\mathbb{R}^d, \gamma^d)^*$, given in 21a2. The freedom in the choice of an orthonormal basis in G is exactly the freedom in the choice of a linear isometry $G \to (\mathbb{R}^d)^*$.

21b Toward Gaussian random processes

Let $G \subset L_2(\Omega, P)$ be a d-dimensional Gaussian space, $g_1, \dots, g_n \in G$ some vectors that span G.[2] As we know, there exists a measure preserving map $\alpha : (\Omega, P) \to (\mathbb{R}^d, \gamma^d)$ such that $\tilde\alpha$ establishes a linear isometry of $(\mathbb{R}^d)^*$ onto G. Taking $f_1, \dots, f_n \in (\mathbb{R}^d)^*$ such that $\tilde\alpha(f_1) = g_1, \dots, \tilde\alpha(f_n) = g_n$ we see that the joint distribution of $f_1, \dots, f_n$ is equal to the joint distribution of $g_1, \dots, g_n$. In this sense, $(g_1, \dots, g_n)$ is a copy of $(f_1, \dots, f_n)$.

The freedom in the choice of a linear isometry between G and $(\mathbb{R}^d)^*$ is exactly the freedom in the choice of $(f_1, \dots, f_n)$ isometric to $(g_1, \dots, g_n)$ in the sense that[3]

(21b1) $$\langle f_i, f_j \rangle = \langle g_i, g_j \rangle \quad \text{for all } i, j \in \{1, \dots, n\} .$$

Indeed, having such $f_1, \dots, f_n$ we may take $\tilde\alpha(c_1 f_1 + \dots + c_n f_n) = c_1 g_1 + \dots + c_n g_n$ for all $c_1, \dots, c_n \in \mathbb{R}$. We see that two Gaussian random vectors are isometric if and only if they are identically distributed.

All said above holds for arbitrary families $(g_i)_{i \in I}$, $(f_i)_{i \in I}$, $g_i \in G$, $f_i \in (\mathbb{R}^d)^*$. (Note that I may be infinite, but d is still finite.) If they are isometric then $g_i = \tilde\alpha(f_i)$, that is,

(21b2) $$\forall i \in I \;\; g_i(\omega) = f_i(\alpha(\omega))$$

for some measure preserving $\alpha : (\Omega, P) \to (\mathbb{R}^d, \gamma^d)$. By "identically distributed" we mean here that $(f_{i_1}, \dots, f_{i_n})$ and $(f_{i_1}, \dots, f_{i_n})$ are identically distributed for all n and $i_1, \dots, i_n \in I$.

---

[1] Once again, more exactly: equivalence class of maps.

[2] Note that n may exceed d. Moreover, a bit later n will become infinite, and even a continuum.

[3] Usually, "isometric" means rather $|f_i - f_j| = |g_i - g_j|$. Then, (21b1) means that $(0, f_1, \dots, f_n)$ is isometric to $(0, g_1, \dots, g_n)$. The two interpretations of "isometric" coincide in the special case $|f_i| = 1$, $|g_i| = 1$, since in this case $|f_i - f_j|^2 = 2 - 2\langle f_i, f_j \rangle$.

21b3 Example. Take $I = \mathbb{R}$, and (using $t \in \mathbb{R}$ instead of $i \in I$ and $2N$ instead of d) define $f_t \in (\mathbb{R}^{2N})^*$ by

$$f_t(x_1, \dots, x_{2N}) = \sum_{k=1}^N a_k \operatorname{Re}\big((x_{2k-1} + \mathrm{i}x_{2k})\mathrm{e}^{\mathrm{i}\lambda_k t}\big) = \\ = \sum_{k=1}^N a_k (x_{2k-1}\cos\lambda_k t - x_{2k}\sin\lambda_k t)$$

where a_k and λ_k are as in Sect. 2. Then $\langle f_s, f_t\rangle = \langle f_0, f_{t-s}\rangle$ (stationarity), and

$$\langle f_0, f_t\rangle = \mathbb{E}\,(f_0 f_t) = \int_{\mathbb{R}^{2N}} f_0 f_t \,\mathrm{d}\gamma^{2N} = \sum_{k=1}^N a_k^2 \cos\lambda_k t\,.$$

The random function $t \mapsto f_t(x)$, where x runs over the probability space $(\mathbb{R}^{2N}, \gamma^{2N})$, is just the random trigonometric sum examined in Sect. 2 (and 11–13).

21c Random processes versus random functions

21c1 Example. Assume that $G \subset L_2(\Omega, P)$ is a $2N$-dimensional Gaussian space, and $(g_t)_{t\in\mathbb{R}}$ a family of $g_t \in G$ such that $\langle g_s, g_t\rangle = \langle g_0, g_{t-s}\rangle$ and

$$\langle g_0, g_t\rangle = \sum_{k=1}^N a_k^2 \cos\lambda_k t\,.$$

Then there exists a measure preserving $\alpha : (\Omega, P) \to (\mathbb{R}^{2N}, \gamma^{2N})$ such that $\tilde\alpha(f_t) = g_t$ for all t. Does it mean that results of Sect. 2 hold for the random function $t \mapsto g_t(\omega)$, since it is a copy of the random trigonometric sum of Sect. 2?

Do not answer in hurry "yes" or "no"; look closely at the question. It is a nonsense! There is no function $t \mapsto g_t(\omega)$, since each g_t is an equivalence class, not a function. Roughly speaking, $g_t(\omega)$ is undefined on a set $A_t \subset \Omega$ negligible in the sense that $P(A_t) = 0$; thus the whole function $t \mapsto g_t(\omega)$ is undefined on the set $\cup_{t\in\mathbb{R}} A_t$ that need not be negligible.

Regretfully, there is no universally agreed definition of a random process. Surely, a random process on a set I is either a function $X : I \times \Omega \to \mathbb{R}$ such that $X(i, \cdot)$ is measurable for each $i \in I$, or an equivalence class of such functions. However, two very different equivalence relations suggest themselves; one is

$$X \sim Y \quad\Longleftrightarrow\quad \forall i \in I \;\; \mathbb{P}\big(\{\omega \in \Omega : X(i,\omega) = Y(i,\omega)\}\big) = 1\,,$$

the other is

$$X \sim Y \iff \mathbb{P}\big(\{\omega \in \Omega : \forall i \in I \;\; X(i,\omega) = Y(i,\omega)\}\big) = 1\,.$$

Let us call their equivalence classes "broad" and "narrow" respectively. Also, let us call a broad equivalence class a *random process,* and a narrow equivalence class — a *random function.*

In other words, a random process is a map $\Xi : I \to L_0(\Omega, P)$ (from I to equivalence classes of measurable functions on Ω), while a random function is an equivalence class of maps $\xi : \Omega \to \mathbb{R}^I$ (from Ω to functions on I) such that $\xi(\cdot)(i)$ is measurable for each $i \in I$; here equivalence is just a.s. equality (of functions on I),

$$\begin{aligned}\xi_1 \sim \xi_2 \quad \iff \quad \mathbb{P}\big(\xi_1 = \xi_2\big) &= \mathbb{P}\big(\{\omega \in \Omega : \xi_1(\omega) = \xi_2(\omega)\}\big) = \\ &= \mathbb{P}\big(\{\omega \in \Omega : \forall i \in I \;\; \xi_1(\omega)(i) = \xi_2(\omega)(i)\}\big) = 1\,.\end{aligned}$$

We say that a random function ξ is a *modification* of a random process Ξ if the narrow equivalence class of ξ is contained in the broad equivalence class of Ξ. In other words: if for each $i \in I$ the function $\xi(\cdot)(i)$ belongs to the equivalence class $\Xi(i)$.[1]

21c2 Exercise. (a) Every random process has a modification (at least one).

(b) If Ω is (finite or) countable then every random process has only one modification.

(c) If I is (finite or) countable then every random process has only one modification.

(d) If I is of cardinality continuum and (Ω, P) is a (complete) nonatomic probability space of cardinality continuum, then every random process has more than one modification.

Prove it.

Returning to 21c1 we see now that $(g_t)_{t\in\mathbb{R}}$ is a random process, not a random function. In contrast, 21b3 gives not only a random process $(f_t)_{t\in\mathbb{R}}$ but also its modification, a random function $f : (\mathbb{R}^{2N}, \gamma^{2N}) \to \mathbb{R}^{\mathbb{R}}$, $f(x_1, \dots, x_{2N})(t) = f_t(x_1, \dots, x_{2N})$.

21c3 Exercise. Some modification g of the random process $(g_t)_{t\in\mathbb{R}}$ of 21c1 is a copy of the random function f of 21b3. In other words, there exists a measure preserving map $\alpha : (\Omega, P) \to (\mathbb{R}^{2N}, \gamma^{2N})$ such that for almost all $\omega \in \Omega$,

$$\forall t \in \mathbb{R} \quad g(\omega)(t) = f\big(\alpha(\omega)\big)(t)\,.$$

Prove it.

[1] Can we say "the function $\xi(\cdot)(i)$"? Yes, it is harmless (think, why) in this context, and in all reasonable contexts.

Thus, results of Sect. 2 hold for some modification of the random process of 21c1.

Continuity of a random function $\xi : \Omega \to \mathbb{R}^I$ can be defined when $I = \mathbb{R}$ or, more generally, $I = (I, \rho)$ is a metric space. Namely, ξ is called (sample) continuous, if for almost all ω, $\xi(\omega)$ is a continuous function (on I).

21c4 Exercise. A random process on $\mathbb{R}$ has no more than one continuous modification.

Prove it.

The same holds for every *separable* metric space (I, ρ).[1]

Now we see that "some modification" in 21c3 may be replaced with "the continuous modification". By the way: many random processes (martingales, processes with independent increments etc.) have (among others) the right-continuous modification and the left-continuous modification (generally different), which is not the case for (centered) Gaussian processes.

Sample continuity should not be confused with continuity in probability.[2] A random process $\Xi : \mathbb{R} \to L_0(\Omega, P)$ is called continuous in probability, if $\Xi(t_n) \to \Xi(t)$ in probability whenever $t_n \to t$. Likewise, $\Xi : \mathbb{R} \to L_2(\Omega, P)$ is called mean-square continuous, if $\Xi(t_n) \to \Xi(t)$ in L_2 whenever $t_n \to t$. (Any metric space may be used here instead of $\mathbb{R}$.) For a Gaussian process $\Xi : \mathbb{R} \to G \subset L_2(\Omega, P)$ continuity in probability is equivalent to mean-square continuity (think, why).

21c5 Exercise. Let $(\Omega, P) = (\mathbb{R}, \gamma^1)$. Define a process $\Xi : \mathbb{R} \to L_2(\Omega, p)$ by $\Xi(s)(t) = \operatorname{sgn}(t - s)$. Prove that Ξ is mean-square continuous but has no sample continuous modification.

21c6 Exercise. Let $G \subset L_2(\Omega, P)$ be a finite-dimensional Gaussian space, and $\Xi : \mathbb{R} \to G$. Then Ξ is sample continuous if and only if it is mean-square continuous.

Prove it.

21d Gaussian spaces: infinite dimension

If $X_n \to X$ in $L_2(\Omega, P)$ (or just in probability) then $X_n \to X$ in distribution, that is,

$$\mathbb{E}\, f(X_n) \to \mathbb{E}\, f(X) \quad \text{for every bounded continuous } f : \mathbb{R} \to \mathbb{R}\,.$$

[1] If I is just a set, not a metric space, we may seek a modification continuous w.r.t. *some* (at least one) separable metric. Surprisingly, still, there is at most one such modification, — so-called natural modification (B. Tsirelson 1976).

[2] They could be called just "continuity of a modification" and "continuity of a process".

Therefore the set of all $X \in L_2(\Omega, P)$ distributed γ^1 is closed. It follows easily that if a non-closed linear subset $G_0 \subset L_2(\Omega, P)$ satisfies 21a1 then its closure also satisfies it and so, is a Gaussian space.

Thus, every sequence $g_1, g_2, \dots$ of independent standard normal random variables spans a Gaussian space.

Such a sequence exists even in $L_2(0,1)$ (tricks with binary digits...). The map

$$(0,1) \to \mathbb{R}^\infty , \quad \omega \mapsto \big(g_1(\omega), g_2(\omega), \dots \big)$$

sends the Lebesgue measure on $(0,1)$ to the so-called standard Gaussian measure γ^∞ on the space $\mathbb{R}^\infty$ of all sequences of reals (with the σ-field generated by the coordinates). It is the infinite product,

$$\gamma^\infty = \gamma^1 \times \gamma^1 \times \dots$$

The linear space $\mathbb{R}^\infty$ is dual to the linear space $\mathbb{R}_\infty$ of all finitely supported sequences of reals:

$$\langle c, x \rangle = \sum_{k=1}^\infty c_k x_k \quad \text{for } c \in \mathbb{R}_\infty,\ x \in \mathbb{R}^\infty .$$

Thus, $c \in \mathbb{R}_\infty$ may be thought of as a measurable linear functional on $(\mathbb{R}^\infty, \gamma^\infty)$, whose norm is equal to the l_2 norm of c,

$$\int_{\mathbb{R}^\infty} (c_1 x_1 + c_2 x_2 + \dots)^2 \, \gamma^\infty(\mathrm{d}x) = c_1^2 + c_2^2 + \dots \quad \text{for } c \in \mathbb{R}_\infty .$$

Such functionals are a non-closed linear subset of $L_2(\mathbb{R}^\infty, \gamma^\infty)$ satisfying 21a1; therefore its closure is an infinite-dimensional Gaussian space $(\mathbb{R}^\infty, \gamma^\infty)^*$. Its elements are called measurable linear functionals on $(\mathbb{R}^\infty, \gamma^\infty)$. Treating $\mathbb{R}_\infty$ as a non-closed linear subset of l_2 we have a linear isometric embedding $\mathbb{R}_\infty \to (\mathbb{R}^\infty, \gamma^\infty)^*$; it extends by continuity to a linear isometric map $l_2 \to (\mathbb{R}^\infty, \gamma^\infty)^*$; this map is onto; thus, roughly speaking,

$$(\mathbb{R}^\infty, \gamma^\infty)^* = l_2 .$$

That is, the general form of a measurable linear functional f on $(\mathbb{R}^\infty, \gamma^\infty)$ is

$$f(x_1, x_2, \dots) = \lim_{n \to \infty} (c_1 x_1 + \dots + c_n x_n) ,$$

the limit being taken in $L_2(\mathbb{R}^\infty, \gamma^\infty)$.[1]

[1] In fact, the series converges almost surely.

In contrast to the finite-dimensional case we have $\gamma^\infty(l_2) = 0$; points of $(\mathbb{R}^\infty, \gamma^\infty)$ cannot be thought of as linear functionals on l_2. In this sense $(\mathbb{R}^\infty, \gamma^\infty)^* = l_2$, but $l_2^* \ne (\mathbb{R}^\infty, \gamma^\infty)$.

Given a Gaussian space $G \subset L_2(\Omega, P)$ and an orthonormal basis $g_1, g_2, \dots \in G$, we get a measure preserving map

$$\alpha : (\Omega, P) \to (\mathbb{R}^\infty, \gamma^\infty) , \quad \alpha(\omega) = \big(g_1(\omega), g_2(\omega), \dots \big)$$

and the corresponding isometric embedding $\tilde\alpha : L_2(\mathbb{R}^\infty, \gamma^\infty) \to L_2(\Omega, P)$ maps $(\mathbb{R}^\infty, \gamma^\infty)^*$ onto G. In this sense G is a copy of $(\mathbb{R}^\infty, \gamma^\infty)^*$.

Similarly to Sect. 21b, every Gaussian random process $\Xi : I \to G \subset L_2(\Omega, P)$ is a copy of a Gaussian random process $\Xi_0 : I \to (\mathbb{R}^\infty, \gamma^\infty)^* \subset L_2(\mathbb{R}^\infty, \gamma^\infty)$, provided that G is separable.[1] It is necessary and sufficient that they are isometric,

$$\mathbb{E}\, \Xi(i_1)\Xi(i_2) = \mathbb{E}\, \Xi_0(i_1)\Xi_0(i_2) \quad \text{for all } i_1, i_2 \in I \, .$$

All infinite-dimensional separable Hilbert spaces are linearly isometric to each other. Thus we may start with an arbitrary map $\Psi : I \to H$ from an arbitrary set I to an arbitrary separable Hilbert space H, and use a linear isometry between H and a Gaussian space G for constructing a Gaussian process $\Xi : I \to G$ isometric to Ψ in the sense that

$$\mathbb{E}\, \Xi(i_1)\Xi(i_2) = \langle \Psi(i_1), \Psi(i_2) \rangle \, .$$

Such Ψ is sometimes called a model of Ξ.

21d1 Example. $H = L_2(0, \infty)$, $I = [0, \infty)$, $\Psi(t) = \mathbf{1}_{(0,t)}$; the corresponding Ξ is the Brownian motion B,

$$\mathbb{E}\, B(s)B(t) = s \quad \text{whenever } 0 \le s \le t < \infty \, .$$

It has a continuous modification, but this fact is not trivial.[2]

21d2 Example. $I = H$ and $\Psi(x) = x$; the corresponding Ξ is the so-called isonormal process,

$$\mathbb{E}\, \Xi(x)\Xi(y) = \langle x, y \rangle \quad \text{for all } x, y \in H \, .$$

It is mean-square continuous, but has no continuous modification.[3]

[1] Usually, $L_2(\Omega, P)$ is separable, and therefore G is.

[2] Interestingly, sample functions of this continuous modification are nowhere differentiable, which is far from being trivial.

[3] It has a linear modification. It has a Borel (measurable) modification. But these two properties are incompatible. In some sense it has no “right” modification (E. Glasner, B. Tsirelson, B. Weiss 2005).

The isonormal process is the mother of all Gaussian processes; they are basically its restrictions to various subsets of H. In particular, $B(t) = \Xi(\mathbf{1}_{(0,t)})$.

21d3 Example. Let $H = L_2(\mathbb{R})$, $I = \mathbb{R}$, $\Psi(t)(s) = \mathrm{e}^{-(t-0.5s)}$ for $0.5s < t$, otherwise 0. The corresponding Ξ is the so-called Ornstein-Uhlenbeck process U,

$$\mathbb{E}\, U(s)U(t) = \mathrm{e}^{-|s-t|} \quad \text{for } s,t \in \mathbb{R}\,.$$

21d4 Exercise. Let B be the Brownian motion, and $X(t) = \mathrm{e}^{-t}B(\mathrm{e}^{2t})$ for $t \in \mathbb{R}$; then X is (isometric to) the Ornstein-Uhlenbeck process.

Check it.

21e Stationary Gaussian processes

21e1 Definition. A Gaussian process $\Xi : \mathbb{R} \to G \subset L_2(\Omega, P)$ is called *stationary,* if

$$\forall s,t \in \mathbb{R} \quad \mathbb{E}\, \Xi(s)\Xi(t) = \mathbb{E}\, \Xi(0)\Xi(t-s)\,.$$

In other words, the shifted process is isometric to the original process.

Its model is a vector-function $\Psi : \mathbb{R} \to H$ (H being a Hilbert space) satisfying

$$\forall s,t \in \mathbb{R} \quad \langle \Psi(s), \Psi(t) \rangle = \langle \Psi(0), \Psi(t-s) \rangle\,.$$

Clearly, $\|\Psi(t)\| = \mathrm{const}$; excluding the trivial case $\|\Psi(t)\| = 0$, we always normalize stationary Gaussian processes:

$$\|\Psi(t)\| = 1\,; \qquad \mathbb{E}\, \Xi^2(t) = 1\,.$$

Such Ψ is determined up to isometry by the corresponding bounded shift-invariant metric on $\mathbb{R}$,

$$\rho_\Psi(s,t) = \|\Psi(s) - \Psi(t)\| = \sqrt{2\bigl(1 - r(s-t)\bigr)}\,,$$

where $r(t) = \mathbb{E}\, \Xi(0)\Xi(t)$ is the correlation function. But do not think that every bounded shift-invariant metric is some ρ_Ψ, For example, the metric $\rho(s,t) = \min(1, |s-t|)$ is not.

The Ornstein-Uhlenbeck process is stationary; the corresponding metric is $\rho(s,t) = 2\bigl(1 - \mathrm{e}^{-|s-t|}\bigr)$.

The correlation function of a stationary process is positive definite (in other words, of positive type); it means that

$$\sum_{i,j} z_i \overline{z}_j r(t_i - t_j) \ge 0$$

for all $n = 1, 2, \dots,$ all $t_1, \dots, t_n \in \mathbb{R}$ and all $z_1, \dots, z_n \in \mathbb{C}$. The proof is immediate: $\mathbb{E}\,|\sum z_i \Xi(t_i)|^2 = \sum z_i \overline{z}_j r(t_i - t_j)$.

21e2 Exercise. The following three conditions on a stationary Gaussian process Ξ are equivalent:

(a) Ξ is mean-square continuous;

(b) the correlation function $t \mapsto \mathbb{E}\,\Xi(0)\Xi(t)$ is continuous;

(c) the metric $(s,t) \mapsto \|\Xi(s) - \Xi(t)\|$ is continuous.

Prove it.

The general form of a continuous positive definite function $r : \mathbb{R} \to \mathbb{C}$ is given by Bochner's theorem:

$$r(t) = \int \mathrm{e}^{\mathrm{i}\lambda t}\,\mu(\mathrm{d}\lambda)$$

for some (positive, finite) measure μ on $\mathbb{R}$. In our case r is real-valued, thus μ is symmetric; $r(0) = 1$, thus μ is a probability measure; it is called the spectral measure;

$$\mathbb{E}\,\Xi(0)\Xi(t) = r(t) = \int \cos\lambda t\,\mu(\mathrm{d}\lambda)$$

for every mean-square continuous stationary Gaussian process Ξ. Recall Sect. 2a; there $r(t) = \sum a_k^2 \cos\lambda_k t$, thus $\mu = \sum a_k^2 \frac12 (\delta_{\lambda_k} + \delta_{-\lambda_k})$, as in (11c5).

The spectral measure gives us a model Ψ of the process Ξ:[1]

$$\Psi : \mathbb{R} \to L_2(\mu)\,, \quad \Psi(t)(\lambda) = \mathrm{e}^{\mathrm{i}\lambda t}\,.$$

Sample continuity of a (mean-square continuous) stationary Gaussian process is a highly nontrivial matter. Here is a necessary and sufficient condition (without proof):[2]

$$\int_0^1 \sqrt{\ln \frac{1}{v(\varepsilon)}}\,\mathrm{d}\varepsilon < \infty\,,$$

where $v(\varepsilon)$ is the Lebesgue measure of $\{t \in [0,1] : \|\Xi(t) - \Xi(0)\| < \varepsilon\}$.

If this condition is violated then for every modification the sample functions are unbounded (from above and below) on every interval.[3]

[1] It is a model in a complex Hilbert space, but all needed scalar products are real, thus we may take the real part of the scalar product and treat $\mathbb{C}$ as a two-dimensional real space…

[2] R.M. Dudley 1967 (sufficiency), X. Fernique 1975 (necessity).

[3] Yu.K. Belyaev 1960. I wonder whether the no-go result of E. Glasner, B. Tsirelson and B. Weiss (2005) has a counterpart for this situation.

21f Hints to exercises

21c2(d): find A_t such that $P(A_t) = 0$ but $\cup_t A_t = \Omega$.

21c4: take a dense countable subset of $\mathbb{R}$.

21c5: again, take a dense countable subset of $\mathbb{R}$.

Index