22 Functions of normal random variables

22a Functions of infinitely many random variables

Recall that $\mathbb{R}^\infty$ is equipped with the σ-field $\mathcal{F}$ generated by the coordinates. Denoting by $\mathcal{F}_n$ the sub-σ-field generated by the first n coordinates we have $\mathcal{F}_n \uparrow \mathcal{F}$ (that is, $\mathcal{F}$ is the least sub-σ-field containing all $\mathcal{F}_n$).

We'll consider the relation $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ in general.

22a1 Exercise. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$ sub-σ-fields. Consider all $A \in \mathcal{F}$ such that there exist $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, \dots$ satisfying[1] $P(A \triangle A_n) \to 0$. Prove that all such A are a sub-σ-field.

22a2 Corollary. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_\infty \subset \mathcal{F}$ sub-σ-fields such that $\mathcal{F}_n \uparrow \mathcal{F}_\infty$. Then for every $A \in \mathcal{F}_\infty$ there exist $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, \dots$ satisfying $P(A \triangle A_n) \to 0$.

22a3 Lemma. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_\infty \subset \mathcal{F}$ sub-σ-fields such that $\mathcal{F}_n \uparrow \mathcal{F}_\infty$. Then[2]

$$L_2(\mathcal{F}_n) \uparrow L_2(\mathcal{F}_\infty)$$

(that is, $L_2(\mathcal{F}_\infty)$ is the least (closed linear) subspace containing all $L_2(\mathcal{F}_n)$).

Proof. Clearly, $L_2(\mathcal{F}_n) \uparrow H \subset L_2(\mathcal{F}_\infty)$; we have to prove that $H = L_2(\mathcal{F}_\infty)$. By 22a2, $\mathbf{1}_A \in H$ for every $A \in \mathcal{F}_\infty$. Linear combinations of such indicators approximate every $f \in L_2(\mathcal{F}_\infty)$. $\square$

The orthogonal projection $L_2(\mathcal{F}) \to L_2(\mathcal{F}_n)$ is, by definition, the operator of conditional expectation, $X \mapsto \mathbb{E}\big(X \,\big|\, \mathcal{F}_n\big)$. It follows from 22a3 that

$$\mathbb{E}\big(X \,\big|\, \mathcal{F}_n\big) \to \mathbb{E}\big(X \,\big|\, \mathcal{F}_\infty\big) \quad \text{in } L_2 \text{ as } n \to \infty \tag{22a4}$$

[1] Here $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

[2] Here $L_2(\mathcal{F}_n) = L_2(\Omega, \mathcal{F}_n, P|_{\mathcal{F}_n})$.

for every $X \in L_2(\mathcal{F})$.[1] In particular, $\mathbb{E}\big(X\,\big|\,\mathcal{F}_n\big) \to X$ for $X \in L_2(\mathcal{F}_\infty)$, and $\mathbb{P}\big(A\,\big|\,\mathcal{F}_n\big) \to \mathbf{1}_A$ for $A \in \mathcal{F}_\infty$. In this sense, a random event occurs gradually!

Conditioning is simple on the product $(\Omega, \mathcal{F}, P) = (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2)$ of two probability spaces. We have two independent sub-σ-fields $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2 \subset \mathcal{F}$,

$$\tilde{\mathcal{F}}_1 = \{ A \times \Omega_2 : A \in \mathcal{F}_1 \}, \quad \tilde{\mathcal{F}}_2 = \{ \Omega_1 \times B : B \in \mathcal{F}_2 \}.$$

We'll see that

$$\mathbb{E}\big(f\,\big|\,\mathcal{F}_1\big)(\omega_1, \omega_2) = \int_{\Omega_2} f(\omega_1, \omega_2')\, P_2(\mathrm{d}\omega_2') \tag{22a5}$$

for all $f \in L_2(\Omega, \mathcal{F}, P)$. Denote $f_1(\omega_1) = \int_{\Omega_2} f(\omega_1, \omega_2)\, P_2(\mathrm{d}\omega_2)$.

22a6 Exercise. Prove that $f_1 \in L_2(\Omega_1, P_1)$ and $\|f_1\| \le \|f\|$.

For every $g \in L_2(\Omega_1, P_1)$ we introduce $\tilde{g} \in L_2(\Omega, P)$ by $\tilde{g}(\omega_1, \omega_2) = g(\omega_1)$.

22a7 Exercise. Prove that $\langle f, \tilde{g} \rangle = \langle f_1, g \rangle$ for all $g \in L_2(\Omega, P_1)$.

Now we are in position to minimize $\|f - \tilde{g}\|$ in g:

$$\begin{aligned} \|f - \tilde{g}\|^2 &= \|f\|^2 - 2\langle f, \tilde{g} \rangle + \|\tilde{g}\|^2 = \\ &= \|f\|^2 - 2\langle f_1, g \rangle + \|g\|^2 = \|f\|^2 + \|g - f_1\|^2 - \|f_1\|^2 ; \end{aligned}$$

this value is minimal when $g = f_1$. It means that $\mathbb{E}\big(f\,\big|\,\mathcal{F}_1\big) = \tilde{f}_1$, which proves (22a5).

The probability space $(\mathbb{R}^\infty, \gamma^\infty)$ is isomorphic to $(\mathbb{R}^n, \gamma^n) \times (\mathbb{R}^\infty, \gamma^\infty)$. By (22a5),

$$\begin{aligned} \mathbb{E}\big(f\,\big|\,\mathcal{F}_n\big)(\omega_1, \omega_2, \dots) &= \\ &= \int_{\mathbb{R}^\infty} f(\omega_1, \dots, \omega_n, \omega_{n+1}', \omega_{n+2}', \dots)\, \gamma^\infty(\mathrm{d}\omega_{n+1}' \mathrm{d}\omega_{n+2}' \dots) \end{aligned} \tag{22a8}$$

for all $f \in L_2(\mathbb{R}^\infty, \gamma^\infty)$. Basically,

$$\mathbb{E}\big(f\,\big|\,\mathcal{F}_n\big)(\omega_1, \dots, \omega_n) = \int f(\omega_1, \omega_2, \dots)\, \gamma^\infty(\mathrm{d}\omega_{n+1} \mathrm{d}\omega_{n+2} \dots).$$

[1] In fact, almost sure convergence also holds (the martingale convergence...).

22b The Cameron-Martin formula

The shift $S_a : \mathbb{R} \to \mathbb{R}$, $S_a(x) = x + a$, sends γ^1 to a measure $S_a[\gamma^1]$ with the density $x \mapsto \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(x-a)^2/2} = \mathrm{e}^{ax-a^2/2} \cdot \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$; thus,

$$\frac{\mathrm{d}S_a[\gamma^1]}{\mathrm{d}\gamma^1}(x) = \mathrm{e}^{ax-a^2/2} \,.$$

The corresponding finite-dimensional formula is

$$\frac{\mathrm{d}S_a[\gamma^d]}{\mathrm{d}\gamma^d}(x) = \mathrm{e}^{\langle a,x\rangle - |a|^2/2} \,.$$

That is,

$$\int_{\mathbb{R}^d} f(x+a)\gamma^d(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\mathrm{e}^{\langle a,x\rangle - |a|^2/2}\, \gamma^d(\mathrm{d}x)$$

for every bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$. What about infinite dimension?

22b1 Exercise. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $f, f_n \in L_2(\Omega, P)$, $f_n \to f$ in probability, and $\sup_n \|f_n\|_{L_2} < \infty$.
(a) Prove that $f_n \to f$ in L_1.
(b) Show by example that convergence in L_2 need not hold.
(c) Assuming only $f \in L_0(\Omega, P)$ prove that $f \in L_2(\Omega, P)$.

22b2 Exercise. Let $a \in l_2$, $g(x) = \langle a, x\rangle - \|a\|^2/2$ and $g_n(x) = a_1x_1 + \dots + a_nx_n - (a_1^2 + \dots + a_n^2)/2$. Prove that $\mathrm{e}^{g_n} \to \mathrm{e}^g$ in $L_1(\mathbb{R}^\infty, \gamma^\infty)$.

Thus,

$$\int \mathrm{e}^{\langle a,x\rangle - \|a\|^2/2}\, \gamma^\infty(\mathrm{d}x) = 1 \quad \text{for all } a \in l_2 \,.$$

22b3 Proposition. For every $a \in l_2$,

$$\frac{\mathrm{d}S_a[\gamma^\infty]}{\mathrm{d}\gamma^\infty}(x) = \mathrm{e}^{\langle a,x\rangle - \|a\|^2/2} \,;$$

that is,

$$\int_{\mathbb{R}^\infty} f(x+a)\,\gamma^\infty(\mathrm{d}x) = \int_{\mathbb{R}^\infty} f(x)\mathrm{e}^{\langle a,x\rangle - \|a\|^2/2}\,\gamma^\infty(\mathrm{d}x)$$

for every bounded measurable $f : \mathbb{R}^\infty \to \mathbb{R}$.

Proof. For $f(x_1, x_2, \dots) = \mathbf{1}_{[u_1,v_1]}(x_1) \dots \mathbf{1}_{[u_n,v_n]}(x_n)$ we have on one hand

$$\int f(x+a)\,\gamma^\infty(\mathrm{d}x) = \int \mathbf{1}_{[u_1-a_1,v_1-a_1]}(x_1) \dots \mathbf{1}_{[u_n-a_n,v_n-a_n]}(x_n)\,\gamma^\infty(\mathrm{d}x) = \\ = \gamma^1\big([u_1-a_1, v_1-a_1]\big) \dots \gamma^1\big([u_n-a_n, v_n-a_n]\big)$$

and on the other hand,

$$\int f(x) \mathrm{e}^{\langle a,x\rangle - \|a\|^2/2} \, \gamma^\infty(\mathrm{d}x) =$$
$$= \int \mathbf{1}_{[u_1,v_1]}(x_1) \dots \mathbf{1}_{[u_n,v_n]}(x_n) \mathrm{e}^{a_1x_1+\dots+a_nx_n-(a_1^2+\dots+a_n^2)/2} \, \gamma^n(\mathrm{d}x) \cdot$$
$$\cdot \int \mathrm{e}^{a_{n+1}x_{n+1}+a_{n+2}x_{n+2}+\dots-(a_{n+1}^2+a_{n+2}^2+\dots)/2} \, \gamma^\infty(\mathrm{d}x_{n+1}\mathrm{d}x_{n+2}\dots) =$$
$$= \Big(\int_{u_1}^{v_1} \mathrm{e}^{a_1x_1-a_1^2/2} \, \gamma^1(\mathrm{d}x_1) \Big) \dots \Big(\int_{u_n}^{v_n} \mathrm{e}^{a_nx_n-a_n^2/2} \, \gamma^1(\mathrm{d}x_n) \Big) \cdot 1 \, ,$$

which is the same. Thus, the two measures coincide on a generating algebra of sets. □

We see that the shifted measure $S_a[\gamma^\infty]$ is equivalent (that is, mutually absolutely continuous) to γ^∞, provided that $a \in l_2$.[1] In this sense, vectors of l_2 are admissible shifts for γ^∞.

22b4 Exercise. If $E \subset (\mathbb{R}^\infty, \gamma^\infty)$ is a linear subspace[2] of full measure[3] then $E \supset l_2$.[4]

Prove it.

22c Lipschitz functions

22c1 Definition. A $\mathrm{Lip}(\sigma)$ function on $(\mathbb{R}^\infty, \gamma^\infty)$ (for a given $\sigma \in [0,\infty)$) is $\xi \in L_0(\mathbb{R}^\infty, \gamma^\infty)$ such that for every $a \in l_2$,[5]

$$|\xi(x+a) - \xi(x)| \le \sigma \|a\| \quad \text{for almost all } x \, .$$

Note that the null set of bad x may depend on a.

Clearly, linear functions $x \mapsto \langle a, x \rangle$ for $a \in l_2$ are $\mathrm{Lip}(\|a\|)$. If $\xi_1, \xi_2, \dots \in \mathrm{Lip}(\sigma)$ and $\sup_n \xi_n = \xi < \infty$ a.s. then $\xi \in \mathrm{Lip}(\sigma)$.

It may seem that 22c1 is ridiculously weak. Even a much stronger condition $\forall a \in l_2 \;\; \forall x \in \mathbb{R}^\infty \;\; \xi(x+a) - \xi(x) = 0$ is satisfied by many nonconstant functions! However, w.r.t. γ^∞ they all are either nonmeasurable or constant almost everywhere.

[1] In fact, for $a \notin l_2$ these two measures are singular.

[2] Just linear, not required to be closed in some topology.

[3] That is, $\gamma^\infty(E) = 1$; E need not be Borel, rather, it must contain a Borel set of full measure.

[4] In fact, l_2 is exactly the intersection of all such E. (Think about $E = \{x : \sum a_k x_k \text{ converges }\}$ where a runs over l_2.)

[5] In fact, this condition may be checked only for a dense subset of the unit ball of l_2.

Similarly to 22c1 we may define the $\mathrm{Lip}(\sigma)$ property for an equivalence class $\xi \in L_0(\mathbb{R}^n)$ as follows: for every $a \in \mathbb{R}^n$,

$$|\xi(x+a) - \xi(x)| \le \sigma|a| \quad \text{for almost all } x \in \mathbb{R}^n \,.$$

We'll see that such equivalence class contains a $\mathrm{Lip}(\sigma)$ function. Consider first the one-dimensional case.

Given $f \in L_2(\mathbb{R})$ and $\varepsilon > 0$, we define $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$f_\varepsilon(x) = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(t)\,\mathrm{d}t \,.$$

Note that f_ε is continuous, and $\|f_\varepsilon\|_{L_2} \le \|f\|_{L_2}$.[1] We have

$$\|f_\varepsilon - f\| \to 0 \quad \text{as } \varepsilon \to 0$$

for all $f \in L_2(\mathbb{R})$, since the set of such f is closed in $L_2(\mathbb{R})$ (think, why) and contains all compactly supported Lipschitz functions.

If an equivalence class $f \in L_2(\mathbb{R})$ satisfies $\mathrm{Lip}(\sigma)$ then for every ε the function f_ε satisfies $\mathrm{Lip}(\sigma)$. It follows (via Cauchy sequences) that f_ε converge uniformly as $\varepsilon \to 0$ (think, why); their limit is a $\mathrm{Lip}(\sigma)$ function in the equivalence class f.

The same holds for $\mathrm{Lip}(\sigma)$ functions of L_0.

A similar argument works in $\mathbb{R}^n$. In this sense, the two definitions of $\mathrm{Lip}(\sigma)$ conform in finite dimension. It follows that Theorem 1a2 applies to every $\xi \in L_2(\mathbb{R}^n, \gamma^n)$ satisfying $\mathrm{Lip}(\sigma)$; it gives $\xi[\gamma^n] = f[\gamma^1]$ for an increasing $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathrm{Lip}(\sigma)$; that is, $\xi \in \mathrm{GaussLip}(\sigma)$ as defined in Sect. 11b.

22c2 Exercise. If $\xi \in L_2(\mathbb{R}^\infty, \gamma^\infty)$ satisfies $\mathrm{Lip}(\sigma)$ then for every n, $\xi_n \in L_2(\mathbb{R}^n, \gamma^n)$ defined by $\mathbb{E}\big(\xi \,\big|\, \mathcal{F}_n\big)(x_1, x_2, \dots) = \xi_n(x_1, \dots, x_n)$ satisfies $\mathrm{Lip}(\sigma)$ and therefore belongs to $\mathrm{GaussLip}(\sigma)$.

Prove it.

22c3 Exercise. Every $\xi \in L_2(\mathbb{R}^\infty, \gamma^\infty)$ satisfying $\mathrm{Lip}(\sigma)$ belongs to $\mathrm{GaussLip}(\sigma)$.

Prove it.

22c4 Exercise. Generalize 22c3 to $\xi \in L_0(\mathbb{R}^\infty, \gamma^\infty)$.

[1]Since $|f_\varepsilon(x)|^2 \le \frac{1}{\varepsilon} \int_x^{x+\varepsilon} |f(t)|^2\,\mathrm{d}t$.

22d A special case

In this section, 11d3 is generalized to all mean-square continuous stationary Gaussian processes (even those having no sample continuous modification).

Let $\Xi : [0,1] \to G \subset L_2(\Omega, P)$ be a Gaussian process. Assume that Ξ is measurable (as a map $[0,1] \to G$;[1] mean-square continuity is evidently sufficient but not necessary[2]). Assume also that $\dim G = \infty$; otherwise the matter becomes trivial. Choosing an orthonormal basis $(g_1, g_2, \dots)$ of G we get

$$\Xi(t) = f_1(t)g_1 + f_2(t)g_2 + \dots$$

where $f_k(t) = \langle \Xi(t), g_k \rangle$ (they are measurable); the series converges in $L_2(\Omega, P)$ for each $t \in [0,1]$.

Here is a general fact.

22d1 Lemma. Let $(\Omega, P) = (\Omega_1, P_1) \times (\Omega_2, P_2)$ be the product of two probability spaces, and $f_n : \Omega \to \mathbb{R}$ measurable functions. If the sequence of functions $(f_n(\omega_1, \cdot))_n$ on Ω_2 converges in probability for almost all $\omega_1 \in \Omega_1$ then there exists a measurable function $f : \Omega \to \mathbb{R}$ such that $f_n(\omega_1, \cdot) \to f(\omega_1, \cdot)$ in probability for almost all $\omega_1 \in \Omega_1$.

Remark. Do not think that the convergence itself ensures measurability of f. Such f may be changed on any set A of the form $\cup_{\omega_1 \in \Omega_1} \big(\{\omega_1\} \times A_{\omega_1} \big)$ where each $A_{\omega_1} \subset \Omega_2$ is a null set. Such A is a null set (by Fubini) *provided* that it is measurable; however, it need not be measurable!

Proof. We assume that $f_n : \Omega \to (-1,1)$ (otherwise take $\frac{2}{\pi} \arctan f_n$).

The function $\omega_1 \mapsto \| \lim_n f_n(\omega_1, \cdot) \|_{L_1(\Omega_2)}$ is measurable (on Ω_1), since it is equal to $\lim_n \| f_n(\omega_1, \cdot) \|_{L_1(\Omega_2)}$. Similarly, functions

$$g_n(\omega_1) = \| f_n(\omega_1, \cdot) - \lim_k f_k(\omega_1, \cdot) \|_{L_1(\Omega_2)}$$

are measurable. Also, $g_n \to 0$ a.s.

We assume that $\sum_n \| g_n \|_{L_1(\Omega_1)} < \infty$ (otherwise choose a subsequence).

We have $\| f_n - f_m \|_{L_1(\Omega)} \le \| g_n \|_{L_1(\Omega_1)} + \| g_m \|_{L_1(\Omega_1)}$ (Fubini, and the triangle inequality). Thus $\sum_n \| f_{n+1} - f_n \|_{L_1(\Omega)} < \infty$, which ensures convergence: $f_n \to f$ a.s. on Ω (for some f). Finally, for almost every $\omega_1 \in \Omega_1$ we get $f_n(\omega_1, \cdot) \to f(\omega_1, \cdot)$ a.s. on Ω_2. $\square$

Returning to the Gaussian process, we apply Lemma 22d1 to the sequence of functions $\Xi_n(t, \omega) = f_1(t) g_1(\omega) + \dots + f_n(t) g_n(\omega)$ on $[0,1] \times \Omega$ and get a

[1] Weakly or strongly, it is all the same...

[2] In fact, for *stationary* Gaussian processes it is also necessary.

measurable function $\tilde{\Xi} : [0,1]\times\Omega \to \mathbb{R}$ such that $\Xi_n(t,\cdot) \to \tilde{\Xi}(t,\cdot)$ in $L_2(\Omega, P)$ for each $t \in [0,1]$ (not only in probability for almost all t, since convergence in L_2 is given for all t, and we may change $\tilde{\Xi}$ on the null set of bad t).

Note that we did not find "the right modification" of Ξ. Indeed, $\tilde{\Xi}$ may be changed on a *measurable* set A of the form $\cup_{t\in[0,1]}\big(\{t\} \times A_t\big)$ where each $A_t \subset \Omega$ is a null set. Measurability of A does not imply that $\cup_t A_t$ is a null set. Not all modifications are jointly measurable (as defined below), but many of them are. A jointly measurable modification of a measurable Gaussian process exists,[1] but usually is highly non-unique.

22d2 Definition. A random function $\xi : \Omega \to \mathbb{R}^{[0,1]}$ is *jointly measurable,* if the function $(t,\omega) \mapsto \xi(\omega)(t)$ is measurable on $[0,1] \times \Omega$.

Note that sample functions of such ξ are measurable on $[0,1]$ (which is not sufficient, however).

22d3 Exercise. A sample continuous random function on $\mathbb{R}$ is jointly measurable.

Prove it.

22d4 Exercise. Let ξ_1, ξ_2 be two jointly measurable modifications of the same random process. Then almost all ω satisfy

$$\xi_1(\omega)(\cdot) = \xi_2(\omega)(\cdot) \quad \text{almost everywhere on } [0,1]\,.$$

Prove it.

We did not upgrade Ξ to "the right random function", but we did upgrade it to "the right random element of $L_0([0,1])$".

Given a bounded continuous (or just Borel) function $\varphi : \mathbb{R} \to \mathbb{R}$, we may consider the random variable

$$\int_0^1 \varphi\big(\xi(\cdot,t)\big)\,\mathrm{d}t \in L_0(\Omega)\,;$$

it does not depend on the choice of a jointly measurable modification ξ of the process Ξ, thus, it does not harm to write

$$\int_0^1 \varphi\big(\Xi(t)\big)\,\mathrm{d}t \in L_0(\Omega)\,.$$

[1]In fact, this existence holds for arbitrary (not just Gaussian) measurable processes, and for arbitrary measure spaces in place of $[0,1]$. Also, existence of a jointly measurable modification implies measurability of Ξ (try Fubini. . .).

22d5 Proposition. If a measurable Gaussian process Ξ on $[0,1]$ satisfies $\|\Xi(t)\| \le 1$ for almost all $t \in [0,1]$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lip(1) function, then the random variable $\int_0^1 \varphi\big(\Xi(t)\big)\,\mathrm{d}t$ belongs to GaussLip(1).

Proof. We assume that $(\Omega, P) = (\mathbb{R}^\infty, \gamma^\infty)$ and $G = (\mathbb{R}^\infty, \gamma^\infty)^*$ (otherwise, use a measure preserving map...). By 22c3 it is sufficient to prove that $\int \varphi\big(\Xi(t)\big)\,\mathrm{d}t$ is a Lip(1) function on $(\mathbb{R}^\infty, \gamma^\infty)$.

Each element g of $G = l_2$, being a linear functional on $(\mathbb{R}^\infty, \gamma^\infty)$, is $\operatorname{Lip}(\|g\|)$. In particular, $\Xi(t) \in \operatorname{Lip}(1)$ for all $t \in [0,1]$; that is, $|\Xi(t)(\cdot + a) - \Xi(t)(\cdot)| \le \|a\|$ a.s. (the null set may depend on a and t). In terms of a jointly measurable modification ξ of the process Ξ,

$$|\xi(x+a)(t) - \xi(x)(t)| \le \|a\| \quad \text{for almost all } (t,x) \in [0,1] \times (\mathbb{R}^\infty, \gamma^\infty)\,.$$

Therefore

$$\begin{aligned}\Big| \int \varphi\big(\xi(x+a)(t)\big)\,\mathrm{d}t - \int \varphi\big(\xi(x)(t)\big)\,\mathrm{d}t \Big| &\le \\ \le \int \big|\varphi\big(\xi(x+a)(t)\big) - \varphi\big(\xi(x)(t)\big)\big|\,\mathrm{d}t &\le \\ \le \int |\xi(x+a)(t) - \xi(x)(t)|\,\mathrm{d}t &\le \|a\|\end{aligned}$$

for almost all $x \in (\mathbb{R}^\infty, \gamma^\infty)$. □

22e Hints to exercises

22a6: $|f_1(\omega_1)|^2 \le \int |f(\omega_1,\omega_2)|^2 \, P_2(\mathrm{d}\omega_2)$.

22a7: try $\int \big(\int \dots P_2(\mathrm{d}\omega_2) \big) \, P_1(\mathrm{d}\omega_1)$.

22b1: (a) On a set of small measure, L_1 norm is much less than L_2 norm.

22c2: (22a8), and Fubini.

22c3: 22c2, and 22a4.

22c4: consider $\xi_M = \operatorname{mid}(-M, \xi, M)$.

22d3: $\xi_n(t,\omega) = \xi\big(\frac{k}{n}, \omega\big)$ for $\frac{k}{n} \le t < \frac{k+1}{n}$.

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