24 Random real zeroes: one derivative

24a Derivative

24a1 Definition. A Gaussian process $\Xi : \mathbb{R} \to G \subset L_2(\Omega, P)$ is *mean-square differentiable,* if for every $t \in \mathbb{R}$ the limit

$$\Xi'(t) = \lim_{s \to t, s \ne t} \frac{\Xi(s) - \Xi(t)}{s - t}$$

exists in $L_2(\Omega, P)$.

The derivative $\Xi' : \mathbb{R} \to G$ is another Gaussian process.

24a2 Exercise. If Ξ is mean-square differentiable then Ξ is mean-square continuous, and Ξ' is measurable.
Prove it.

24a3 Exercise. If a stationary Gaussian process Ξ is mean-square differentiable then $\big(\Xi(t+h) - \Xi(t)\big)/h$ converges to $\Xi'(t)$ (in $L_2(\Omega, P)$, as $h \to 0$) uniformly in t.
Prove it.

Thus, stationarity ensures that Ξ' is mean-square continuous.
It was rather about vector-functions; probability enters now.

24a4 Lemma. Let a Gaussian process Ξ be mean-square differentiable and Ξ' mean-square continuous, then for each t,

$$\Xi(t) = \Xi(0) + \int_0^t Y(s) \, \mathrm{d}s \, ,$$

where Y is a jointly measurable modification of Ξ'.

Proof. Let $t = 1$ (the general case is similar). We have (recall 23a)

$$\Xi(t) = \sum_{k=1}^{\infty} f_k(t) g_k \,, \quad f_k(t) = \langle \Xi(t), g_k \rangle \,.$$

Each f_k is continuously differentiable, and

$$\Xi'(t) = \sum_{k=1}^{\infty} f'_k(t) g_k$$

(think, why). We apply 23a2 to Y and $f(\cdot) = 1$:

$$\underbrace{\sum_{k=1}^{\infty} \Big(\int_0^1 f'_k(t) \, \mathrm{d}t \Big) g_k}_{=\sum (f_k(1) - f_k(0)) g_k = \Xi(1) - \Xi(0)} = \int_0^1 Y(t) \, \mathrm{d}t \quad \text{in } L_2(\Omega, P) \,.$$

□

The following conclusion is trivial when Ξ' is sample continuous, but nontrivial in general.

24a5 Proposition. If a stationary Gaussian process Ξ is mean-square differentiable then it has a sample continuous modification[1] X, and

$$\forall t \in \mathbb{R} \quad X(t) = X(0) + \int_0^t Y(s) \, \mathrm{d}s \,,$$

where Y is a jointly measurable modification of Ξ'.

24a6 Exercise. (a) If a Gaussian process has a sample continuous modification then it is mean-square continuous.

(b) If a Gaussian random function is continuously differentiable (almost surely), then it is mean-square continuously differentiable (that is, mean-square differentiable, and the derivative is mean-square continuous).

Prove it. (Stationarity is not assumed.)

24b Rice's formula

The proof of Theorem 2b1 (and in particular Rice's formula) given in Sect. 12a for finite dimension, generalizes easily to stationary processes with sample

[1] This claim also follows easily from the criterion at the end of Sect. 21e.

continuous *second* derivative. However, the formula does not involve the second derivative, thus it is natural not to assume its existence. But then it is not evident whether or not (a) $\{t \in [0,1] : X(t) = 0\}$ is finite; (b) $X(\cdot)$ is piecewise monotone; (c) $X(\cdot)$ and $X'(\cdot)$ cannot vanish simultaneously. And nevertheless Theorem 2b1 generalizes, as follows.

24b1 Theorem. Let a stationary Gaussian random function X be continuously differentiable (almost surely), $\mathbb{E}\, X^2(0) = 1$ and $\mathbb{E}\, X'^2(0) = 1$. Let a measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy $\int |\varphi(y)||y|\mathrm{e}^{-y^2/2}\, \mathrm{d}y < \infty$. Then $\{t \in [0,1] : X(t) = 0\}$ is finite almost surely, the random variable

$$\xi = \sum_{t\in[0,1],X(t)=0} \varphi\bigl(X'(t)\bigr)$$

is integrable, and

$$\mathbb{E}\,\xi = \frac{1}{2\pi} \int \varphi(y)|y|\mathrm{e}^{-y^2/2}\, \mathrm{d}y\,.$$

Similarly to 2b1, $[0,1]$ may be replaced with $[0,L]$.

In particular (for $\varphi(\cdot) = 1$), the expected number of zeroes per unit time is equal to $1/\pi$ (Rice's formula).

The idea of the proof is a discrete approximation of the continuous time. Instead of $X(t)$ for $t \in [0,1]$ we consider $X\bigl(\frac{k}{2^n}\bigr)$ for $k = 0, 1, \dots, 2^n$, and instead of $t \in [0,1]$ such that $X(t) = 0$ we consider $k \in \{1, 2, \dots, 2^n\}$ such that

$$X\Bigl(\frac{k-1}{2^n}\Bigr)X\Bigl(\frac{k}{2^n}\Bigr) < 0\,.$$

Denote by Z_n the (random) number of these k.

24b2 Exercise. Let $G \subset L_2(\Omega, P)$ be a Gaussian space and $g_1, g_2 \in G$, $\|g_1\| = \|g_2\| = 1$. Then

$$\mathbb{P}\bigl(g_1 g_2 < 0\bigr) = \frac{1}{\pi} \arccos\langle g_1, g_2\rangle = \frac{2}{\pi} \arcsin \frac{\|g_1 - g_2\|}{2}\,.$$

Prove it.

24b3 Exercise. Prove that $\mathbb{E}\, Z_n \to 1/\pi$ as $n \to \infty$.

We have $Z_1 \le Z_2 \le \dots$ (think, why) and $\sup_n \mathbb{E}\, Z_n < \infty$, therefore $Z_n \uparrow Z_\infty < \infty$ a.s., and $\mathbb{E}\, Z_\infty = \lim \mathbb{E}\, Z_n = 1/\pi$. It follows easily that

$$\mathbb{E}\,\#\{t \in (0,1) : X(t) = 0, X'(t) \ne 0\} \le 1/\pi\,,$$
$$\mathbb{E}\,\#\{t \in (0,1) : X(t) = 0\} \ge 1/\pi\,.$$

24b4 Lemma. Let $u \in \mathbb{R}$. Almost surely, no $t \in \mathbb{R}$ satisfies both $X(t) = u$ and $X'(t) = 0$.

(The proof will be given later.)

Thus,

$$\mathbb{E}\,\#\{t \in (0,1) : X(t) = 0\} = 1/\pi\,,$$

which proves Theorem 24b1 for $\varphi(\cdot) = 1$ (Rice's formula).

For a measurable $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\int |\varphi(y)||y|\mathrm{e}^{-y^2/2}\,\mathrm{d}y < \infty$, 24b2 generalizes as follows:

$$\mathbb{E}\,\biggl(\varphi\Bigl(\frac{g_1 - g_2}{\|g_1 - g_2\|}\Bigr)\mathbf{1}_{(-\infty,0)}(g_1 g_2)\biggr) = \int_{-\infty}^{+\infty} \gamma^1(\mathrm{d}y)\,\varphi(y)\gamma^1\bigl([-|y|\tan\alpha, |y|\tan\alpha]\bigr)$$

where $\alpha = \frac12 \arccos\langle g_1, g_2\rangle = \arcsin\frac{\|g_1-g_2\|}{2}$. For $\alpha \to 0$ we have $\frac1\alpha\gamma^1\bigl([-|y|\tan\alpha, |y|\tan\alpha]\bigr) \to 2|y|/\sqrt{2\pi}$ and $\frac1\alpha\gamma^1\bigl([-|y|\tan\alpha, |y|\tan\alpha]\bigr) \le 2|y|/\sqrt{2\pi}$, thus

$$\frac1\alpha \int_{-\infty}^{+\infty} \gamma^1(\mathrm{d}y)\,\varphi(y)\gamma^1\bigl([-|y|\tan\alpha, |y|\tan\alpha]\bigr) \to \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \gamma^1(\mathrm{d}y)\,\varphi(y)|y|\,;$$

$$\mathbb{E}\,\biggl(\varphi\Bigl(\frac{g_1 - g_2}{\|g_1 - g_2\|}\Bigr)\mathbf{1}_{(-\infty,0)}(g_1 g_2)\biggr) = \bigl(1 + o(1)\bigr)\frac{\|g_1 - g_2\|}{2\pi} \int_{-\infty}^{+\infty} \varphi(y)|y|\mathrm{e}^{-y^2/2}\,\mathrm{d}y\,.$$

Taking $g_2 = X\bigl(\frac{k-1}{2^n}\bigr)$ and $g_1 = X\bigl(\frac{k}{2^n}\bigr)$ we get an approximation to $\mathbb{E}\,\Bigl(\varphi(X'(t))\,\mathbf{1}_{(-\infty,0)}\bigl(X(\frac{k-1}{2^n})X(\frac{k}{2^n})\bigr)\Bigr)$. That is, we introduce

$$\xi_n = \sum_{k=1}^{2^n} \varphi\biggl(\frac{X(\frac{k}{2^n}) - X(\frac{k-1}{2^n})}{\|X(\frac{k}{2^n}) - X(\frac{k-1}{2^n})\|}\biggr)\mathbf{1}_{(-\infty,0)}\biggl(X\Bigl(\frac{k-1}{2^n}\Bigr)X\Bigl(\frac{k}{2^n}\Bigr)\biggr)$$

and note that $\mathbb{E}\,\xi_n \to \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(y)|y|\mathrm{e}^{-y^2/2}\,\mathrm{d}y$. Assume in addition that $\varphi : \mathbb{R} \to [0,1]$ is continuous. Then $|\xi_n| \le Z_\infty$ a.s., and for every $\varepsilon > 0$,

$$|\xi_n - \xi| \le \varepsilon Z_\infty$$

for all n large enough (that is, $n \ge N(\omega)$); therefore $\xi_n \to \xi$ a.s., and the dominated convergence theorem gives $\mathbb{E}\,\xi_n \to \mathbb{E}\,\xi$.

Theorem 24b1 is thus proved under the additional assumptions on φ. The general case follows, similarly to Sect. 12a (recall 12a4, (12a5)). However, Lemma 24b4 will be proved in the next section.

24c Stratification

A surface in $\mathbb{R}^3$ is negligible (a null set for the three-dimensional Lebesgue measure) since it has negligible intersections with parallel lines. Here is a similar infinite-dimensional argument.[1]

24c1 Lemma. Let a measurable set A in $(\mathbb{R}^\infty, \gamma^\infty)$ be such that

$$\{c \in \mathbb{R} : (x_1 + c, x_2, x_3, \dots) \in A\}$$

is a null set for almost all $(x_1, x_2, x_3, \dots) \in (\mathbb{R}^\infty, \gamma^\infty)$. Then $\gamma^\infty(A) = 0$.

Proof. First,

$$\int \Big(\int f(x+c) \, \mathrm{d}c \Big) \gamma^1(\mathrm{d}x) = \int f(c) \, \mathrm{d}c = \sqrt{2\pi} \int f(x) \mathrm{e}^{x^2/2} \, \gamma^1(\mathrm{d}x) \in [0, \infty]$$

for every measurable $f : \mathbb{R} \to [0, \infty)$. Second,

$$\int \Big(\int f(x_1 + c, x_2, \dots) \, \mathrm{d}c \Big) \gamma^\infty(\mathrm{d}x_1 \mathrm{d}x_2 \dots) = \\ = \sqrt{2\pi} \int f(x_1, x_2, \dots) \mathrm{e}^{x_1^2/2} \, \gamma^\infty(\mathrm{d}x_1 \mathrm{d}x_2 \dots) \in [0, \infty]$$

for every measurable $f : (\mathbb{R}^\infty, \gamma^\infty) \to [0, \infty)$. It remains to apply it to $f = \mathbf{1}_A$. $\square$

24c2 Proposition. Let $\Xi : [0, 1] \to G \subset L_2(\Omega, P)$ be a Gaussian process, as in Sect. 23a:[2]

$$\Xi(t) = f_1(t) g_1 + f_2(t) g_2 + \dots \, , \quad \sum_k \int |f_k(t)|^2 \, \mathrm{d}t < \infty \, ,$$

and $X : \Omega \to L_2[0, 1]$ the corresponding random equivalence class. Let a set $A \subset L_2[0, 1]$ be such that $X^{-1}(A)$ is measurable, and

$$\{c \in \mathbb{R} : X(\cdot) + c f_1 \in A\}$$

is a null set, almost surely. Then $P\big(X^{-1}(A)\big) = 0$.

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[1] This approach is a special case of the "stratification method" developed in the book: Yu.A. Davydov, M.A. Lifshits, N.V. Smorodina, "Local properties of stochastic functionals", AMS 1998 (transl. from Russian 1995).

[2] In fact, we may waive the condition $\sum \int |f_k(t)|^2 \, \mathrm{d}t < \infty$ and work in $L_0[0, 1]$ instead of $L_2[0, 1]$. This general case is reduced to the special case by considering $t \mapsto \Xi(t) / \|\Xi(t)\|$.

Proof. Similarly to the proof of 23a3 we assume that $(\Omega, P) = (\mathbb{R}^\infty, \gamma^\infty)$, g_k are the coordinates, note that

$$X(x_1 + c, x_2, \dots) - X(x_1, x_2, \dots) = cf_1$$

for almost all $x \in (\mathbb{R}^\infty, \gamma^\infty)$, and apply Lemma 24c1 to $X^{-1}(A)$. □

Note that $f_1(t) = \langle \Xi(t), g_1 \rangle$, and g_1 is just a unit vector in G; we may choose it at will.[1]

24c3 Corollary. Let X be a sample continuous Gaussian random function on $[0,1]$, $B \subset C[0,1]$ a Borel set, and $f \in C[0,1]$ be defined by $f(t) = \mathbb{E}\big(gX(t)\big)$ for some g of the Gaussian space. If

$$\{c \in \mathbb{R} : X(\cdot) + cf \in B\}$$

is a null set almost surely, then $X(\cdot) \notin B$ almost surely.

24c4 Lemma. Let $f, \varphi : [0,1] \to \mathbb{R}$ be continuously differentiable, $\forall t \in [0,1] \;\; f(t) \ne 0$, and $u \in \mathbb{R}$. Then for almost every $c \in \mathbb{R}$, no $t \in [0,1]$ satisfies both $\varphi(t) + cf(t) = u$ and $\varphi'(t) + cf'(t) = 0$.

Proof. We assume $u = 0$ (otherwise replace $\varphi(\cdot)$ with $\varphi(\cdot) - u$). If such t exists for a given c, then this c is a critical value of the function $-\varphi(\cdot)/f(\cdot)$, since on one hand $c = -\frac{\varphi(t)}{f(t)}$ and on the other hand

$$\Big(-\frac{\varphi}{f}\Big)'(t) = \frac{-f\varphi' + \varphi f'}{f^2}(t) = -\frac{1}{f(t)}\Big(\varphi'(t) \underbrace{- \frac{\varphi(t)f'(t)}{f(t)}}_{+cf'(t)}\Big) = 0\,.$$

By Sard's theorem, critical points of a continuously differentiable function are a null set. □

24c5 Proposition. Let a Gaussian random function X on $\mathbb{R}$ be continuously differentiable (almost surely), and $\forall t \in \mathbb{R} \;\; \mathbb{E}\, X^2(t) \ne 0$. Let $u \in \mathbb{R}$. Then, almost surely, no $t \in \mathbb{R}$ satisfies both $X(t) = u$ and $X'(t) = 0$.

Proof. By 24a6, X is mean-square continuously differentiable. Thus, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \langle X(t), X(0) \rangle = \mathbb{E}\big(X(0)X(t)\big)$$

[1] In fact, every admissible shift of $X[\gamma^\infty]$ may serve as f_1.

is continuously differentiable. Clearly, $f(0) \neq 0$, thus $f(\cdot) \neq 0$ on some $[-\varepsilon, \varepsilon]$. By 24c4, almost surely, for almost every $c \in \mathbb{R}$, no $t \in [-\varepsilon, \varepsilon]$ satisfies both $X(t) + cf(t) = u$ and $X'(t) + cf'(t) = 0$.

The set B of all continuously differentiable functions x such that $\exists t \in [-\varepsilon, \varepsilon] \; \big(x(t) = u, x'(t) = 0 \big)$ is closed, therefore a Borel set. By 24c3, $X(\cdot) \notin B$ almost surely.

That is, $X(t) = u$ together with $X'(t) = 0$ does not happen (almost surely) in a neighborhood of 0. The same holds in a neighborhood of every point of $\mathbb{R}$. It remains to take a countable subcovering of the given open covering. $\square$

Lemma 24b4 follows immediately, which finalizes the proof of Theorem 24b1.

24d Hints to exercises

24a2: Ξ' is the limit of a sequence of continuous vector-functions.

24b2: consider first the two-dimensional Gaussian space $(\mathbb{R}^2, \gamma^2)^*$.

24b3: $\mathbb{E}\, Z_n$ is the sum of 2^n equal probabilities; also, $\|X(\varepsilon) - X(0)\| \sim \varepsilon$.

24a6: convergence almost sure implies convergence in probability; in the Gaussian case the latter is equivalent to mean-square convergence.