1 Fourier transform as unitary equivalence

Functional analysis treats functions as points in the function space. A useless neologism? Not at all. Rather, a way to use geometric intuition for the benefit of analysis.

Surprisingly, simple geometric symmetries of a special, seemingly not notable two-dimensional surface in Hilbert space lead naturally to the famous Fourier transform (and some other useful things).

1a Introduction

On the finite group $\mathbb{Z}/n\mathbb{Z}$, Fourier transform amounts to the basis of eigenvectors of the shift $U : \mathbb{C}^n \to \mathbb{C}^n$,

$$U(f_0, f_1, \dots, f_{n-2}, f_{n-1}) = (f_1, f_2, \dots, f_{n-1}, f_0)\,.$$

The eigenvalue $\mathrm{e}^{2\pi\mathrm{i}l/n}$ corresponds to the eigenvector

$$\bigl(1,\, \mathrm{e}^{2\pi\mathrm{i}l/n},\, \mathrm{e}^{2\pi\mathrm{i}\cdot 2l/n},\, \dots,\, \mathrm{e}^{2\pi\mathrm{i}\cdot(n-1)l/n}\bigr)\,.$$

Defining unitary $\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{F}(f_0, f_1, \dots, f_{n-1}) = \\ = \frac{1}{\sqrt{n}}\Bigl(\sum_k f_k,\, \sum_k \mathrm{e}^{-2\pi\mathrm{i}k/n} f_k,\, \sum_k \mathrm{e}^{-2\pi\mathrm{i}k\cdot 2/n} f_k,\, \dots,\, \sum_k \mathrm{e}^{-2\pi\mathrm{i}k\cdot(n-1)/n} f_k\Bigr)$$

we get $\mathcal{F}U = V\mathcal{F}$ where V is diagonal,

$$V(f_0, f_1, \dots, f_{n-1}) = \big(f_0,\, \mathrm{e}^{2\pi\mathrm{i}/n} f_1,\, \dots,\, \mathrm{e}^{2\pi\mathrm{i}(n-1)/n} f_{n-1}\big)\,.$$

Thus, $\mathcal{F}$ diagonalizes U,

$$\mathcal{F}U\mathcal{F}^{-1} = V\,.$$

On the compact group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ we have Fourier series,

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{2\pi\mathrm{i}kx}\,,$$
$$c_k = \int_0^1 f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x\,.$$

The unitary operator $\mathcal{F} : L_2(\mathbb{T}) \to l_2(\mathbb{Z})$, $\mathcal{F}f = c$, diagonalizes shifts $U(a) : L_2(\mathbb{T}) \to L_2(\mathbb{T})$, $U(a)f : x \mapsto f(x+a)$; namely,

$$\mathcal{F}U(a)\mathcal{F}^{-1} = V(a)\,,$$
$$V(a) : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})\,, \quad (V(a)c)_k = \mathrm{e}^{2\pi\mathrm{i}ak} c_k\,.$$

On the noncompact group $\mathbb{R}$ the situation is similar in principle, but more complicated technically, since the shifts have continuous spectrum.[1] We'll see that the Fourier transform is a unitary operator $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ that diagonalizes shifts $U_1(a) : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $U_1(a)f : t \mapsto f(t+a)$; namely,

$$\mathcal{F}U_1(a)\mathcal{F}^{-1} = V_1(a)\,,$$
$$V_1(a) : L_2(\mathbb{R}) \to L_2(\mathbb{R})\,, \quad V_1(a)f : t \mapsto \mathrm{e}^{\mathrm{i}bt} f(t)\,.$$

In fact, $\mathcal{F}f : t \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}st} f(s)$ for $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

In order to overcome technical difficulties we'll go round. An orthonormal basis is a family of vectors with special (and extremely simple) scalar products. Quite different family of vectors with special (and still simple) scalar products, the so-called exponential map, will be instrumental.

See also "List of results", "List of formulas", and Index.

1b Exponential map

Two special geometric objects are introduced in this section: a curve $\psi(\cdot)$ in a Hilbert space, and the corresponding curve $w(\cdot)$ on the sphere. Placing appropriately the latter curve in the function space $L_2(\mathbb{R})$ we extend symmetries of the curve to shifts of the functions.

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[1]The function $x \mapsto \mathrm{e}^{\mathrm{i}ax}$ is not an eigenvector, since it does not belong to $L_2(\mathbb{R})$.

Toy model

Consider a map (vector-function, parametrized curve) $\psi : \mathbb{R} \to E$, where E is a Euclidean space, satisfying

$$\langle \psi(x), \psi(y) \rangle = 1 + xy + \frac{1}{2}x^2 y^2 \,.$$

Existence: $\psi(x) = \bigl(1, x, x^2/\sqrt{2}\bigr) \in \mathbb{R}^3$.
Uniqueness: $\| \sum c_k \psi_1(x_k) \| = \| \sum c_k \psi_2(x_k) \|$.
General form: $\psi(x) = e_0 + x e_1 + \frac{x^2}{\sqrt{2}} e_2$ where e_0, e_1, e_2 are orthonormal.
By the way, $e_0 = \psi(0)$, $e_1 = \psi'(0)$, $e_2 = \psi''(0)/\sqrt{2}$.
The subspace spanned by all $\psi(x)$ is 3-dimensional.

Exponential map on $\mathbb{R}$

Consider a map (vector-function) $\psi_0 : \mathbb{R} \to H$, where H is a Hilbert space over $\mathbb{R}$, satisfying

(1b1) $$\langle \psi_0(x), \psi_0(y) \rangle = \exp(xy/2) \,; \quad \langle \psi_0(x\sqrt{2}), \psi_0(y\sqrt{2}) \rangle = \exp(xy) \,.$$

Existence: $\psi_0(x\sqrt{2}) = \bigl(1, x, x^2/\sqrt{2!}, x^3/\sqrt{3!}, \dots\bigr) \in l_2$.
As before: uniqueness;

$$\psi_0(x\sqrt{2}) = \sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k!}} e_k \,;$$

e_k are orthonormal; the power series converges (in norm) for all x; $e_k = \frac{2^{k/2}}{\sqrt{k!}} \psi_0^{(k)}(0)$; the subspace spanned by all $\psi_0(x)$ is infinite-dimensional. Assume it to be the whole H.

Alternatively:

$$w_0(x) = \frac{\psi_0(x)}{\|\psi_0(x)\|} = \mathrm{e}^{-x^2/4} \psi_0(x) \,,$$

(1b2) $$\langle w_0(x), w_0(y) \rangle = \exp(-(x-y)^2/4) \,;$$

the latter is shift-invariant.

Action of shifts: $U_0(a) = U^{(w_0)}(a) \in \mathrm{Unitary}(H)$ for $a \in \mathbb{R}$,

$$U_0(a) w_0(x) = w_0(x-a) \,; \quad U_0(a+b) = U_0(a) U_0(b) \,.$$

Observation: $\int_{-\infty}^{+\infty} \mathrm{e}^{-aq^2} \, \mathrm{d}q = \sqrt{\pi/a}$, thus the vector-function $w_1 : \mathbb{R} \to L_2(\mathbb{R})$,

$$w_1(x\sqrt{2}) : q \mapsto \sqrt[4]{\frac{2a}{\pi}} \mathrm{e}^{-(q\sqrt{a}-x)^2}$$

satisfies (1b2). Indeed,

$$\int_{-\infty}^{+\infty} w_1(x\sqrt{2})(q) w_1(y\sqrt{2})(q)\,\mathrm{d}q = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \exp\bigl(-(q\sqrt{a}-x)^2-(q\sqrt{a}-y)^2\bigr)\,\sqrt{a}\mathrm{d}q =$$

$$= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \exp\bigl(-(q-x)^2-(q-y)^2\bigr)\,\mathrm{d}q =$$

$$= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \exp\Biggl(-2\Bigl(q-\frac{x+y}{2}\Bigr)^2 - \Bigl(x^2+y^2-\frac{(x+y)^2}{2}\Bigr)\Biggr)\,\mathrm{d}q =$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}} \exp\Bigl(-\frac{(x-y)^2}{2}\Bigr).$$

Traditionally one chooses $a = 1/2$; the vector-functions $w_1, \psi_1 : \mathbb{R} \to L_2(\mathbb{R})$,

$$w_1(x) : q \mapsto \pi^{-1/4}\mathrm{e}^{-(q-x)^2/2} = \pi^{-1/4}\exp\Bigl(-\frac{q^2}{2}+xq-\frac{x^2}{2}\Bigr),$$

$$\psi_1(x) : q \mapsto \pi^{-1/4}\exp\Bigl(-\frac{q^2}{2}+xq-\frac{x^2}{4}\Bigr) \tag{1b3}$$

satisfy (1b2), (1b1) respectively. It is not evident whether the vectors $\psi_1(x)$ span $L_2(\mathbb{R})$ or not.[1] But anyway, our shifts $U_1(a) = U^{(w_1)}(a)$ conform to the usual shifts on $L_2(\mathbb{R})$; indeed, $\bigl(U_1(a)w_1(x)\bigr)(q) = w_1(x-a)(q) = w_1(x)(q+a)$, since $w_1(x)(q)$ is a function of $q - x$ only. Thus,

$$U_1(a)f : q \mapsto f(q+a)$$

for every f of the spanned subspace.

1c Exponential map as an analytic vector-function

Using analytic vector-functions on the complex plane we turn to special two-dimensional surfaces in the Hilbert space, thus gaining additional symmetries. In the function space the shifts appear to be diagonalized. Also, some useful systems of functions generate $L_2(\mathbb{R})$.

Now we assume H to be a Hilbert space over $\mathbb{C}$ and note that the unitary operator $e_k \mapsto \mathrm{i}^k e_k$ leads to another vector-function $x\sqrt{2} \mapsto \sum \frac{x^k}{\sqrt{k!}}\mathrm{i}^k e_k$ satisfying (1b1). Not exciting, unless we turn to ψ_1, try

$$\psi_1(\mathrm{i}x) : q \mapsto \pi^{-1/4}\exp\Bigl(-\frac{q^2}{2}+\mathrm{i}xq+\frac{x^2}{4}\Bigr),$$

$$\frac{\psi_1(\mathrm{i}x)}{\|\psi_1(\mathrm{i}x)\|} : q \mapsto \pi^{-1/4}\exp\Bigl(-\frac{q^2}{2}+\mathrm{i}xq\Bigr) = \mathrm{e}^{\mathrm{i}xq}\psi_1(0)(q)$$

[1] We'll see that they do.

and observe that our shifts $U_1(a)$ turn into multiplication operators. Really?!

Consider a vector-function $\psi_0 : \mathbb{C} \to H$ satisfying

$$\langle \psi_0(z_1), \psi_0(z_2) \rangle = \exp(z_1 \overline{z}_2 / 2) \, . \tag{1c1}$$

Existence: $\psi_0(z\sqrt{2}) = \big(1, z, z^2/\sqrt{2!}, z^3/\sqrt{3!}, \dots\big) \in l_2$.

As before: uniqueness;

$$\psi_0(z\sqrt{2}) = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} e_k \, ;$$

e_k are orthonormal; the power series converges (in norm) for all z, which means that ψ_0 is an *entire vector-function*; $e_k = \frac{2^{k/2}}{\sqrt{k!}} \psi_0^{(k)}(0)$; the subspace spanned by all $\psi_0(z)$ is infinite-dimensional, and we assume it to be the whole H.

The vector-function $\mathbb{R} \ni x \mapsto \psi_0(x) \in H$ satisfies (1b1). Another vector-function $\mathbb{R} \ni x \mapsto \psi_0(\mathrm{i}x) \in H$ also satisfies (1b1), since $\mathrm{i}x\overline{\mathrm{i}y} = xy$.

We introduce a vector-function $\psi_1 : \mathbb{C} \to L_2(\mathbb{R})$ by

$$\psi_1(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4} \Big) \tag{1c2}$$

and note that $\langle \psi_1(x), \psi_1(y) \rangle = \exp(xy/2)$ for $x, y \in \mathbb{R}$. Does it mean that $\langle \psi_1(z_1), \psi_1(z_2) \rangle = \exp(z_1 \overline{z}_2/2)$ for $z_1, z_2 \in \mathbb{C}$? This equality could be checked by a calculation, but it is more interesting to get it from an excursion into the theory of analytic vector-functions (which provides much more than just this equality).

Entire vector-functions on $\mathbb{C}$

1c3 Definition. An *entire vector-function* $\varphi : \mathbb{C} \to H$ is a vector-function of the form

$$\varphi(z) = \sum_{k=0}^{\infty} z^k h_k$$

in the sense that

$$\Big\| \varphi(z) - \sum_{k=0}^{n} z^k h_k \Big\| \to 0 \quad \text{as } n \to \infty$$

for every $z \in \mathbb{C}$; here H is a Hilbert space, and $h_k \in H$.

Clearly, h_k must satisfy $\sqrt[k]{\|h_k\|} \to 0$ as $k \to \infty$.

The one-dimensional case $H = \mathbb{C}$ conforms to the usual notion of (scalar) entire function.

1c4 Exercise. If $\varphi(z) = \sum_{k=0}^\infty z^k h_k$ is an entire vector-function then $\xi(z) = \sum_{k=0}^\infty (k+1) z^k h_{k+1}$ is an entire vector-function, and $\varphi' = \xi$ in the sense that for every $z \in \mathbb{C}$,

$$\Big\| \frac{\varphi(z+\Delta z) - \varphi(z)}{\Delta z} - \xi(z) \Big\| \to 0 \quad \text{as } |\Delta z| \to 0+, \ \Delta z \in \mathbb{C} \, .$$

Prove it.

Hint: $(z_1 - z_0) \int_0^1 \xi\big((1-u)z_0 + u z_1\big) \, \mathrm{d}u = \varphi(z_1) - \varphi(z_0)$.

1c5 Corollary. If $\varphi(z) = \sum_{k=0}^\infty z^k h_k$ is an entire vector-function then φ is infinitely differentiable, and

$$h_n = \frac{1}{n!} \varphi^{(n)}(0) \quad \text{for } n = 0, 1, 2, \dots$$

Thus,

$$\varphi(z) = \sum_{k=0}^\infty \frac{z^k}{k!} \varphi^{(k)}(0) \, .$$

If entire vector-functions $\varphi_1, \varphi_2 : \mathbb{C} \to H$ satisfy $\varphi_1^{(n)}(0) = \varphi_2^{(n)}(0)$ for all n, then $\varphi_1 = \varphi_2$.

1c6 Proposition. For arbitrary entire vector-functions $\varphi_1, \varphi_2 : \mathbb{C} \to H$,

(a) if $\varphi_1(x) = \varphi_2(x)$ for all $x \in \mathbb{R}$ then $\varphi_1 = \varphi_2$;

(b) moreover, if the set $\{z \in \mathbb{C} : \varphi_1(z) = \varphi_2(z)\}$ has at least one (finite) accumulation point then $\varphi_1 = \varphi_2$.

1c7 Proposition. For every entire vector-function $\varphi : \mathbb{C} \to H$,

(a) the closed subspace spanned by all $\varphi(z)$ contains all $\varphi'(z)$ and moreover, all $\varphi^{(n)}(z)$;

(b) the closed subspace spanned by all $\varphi(z)$ is equal to the closed subspace spanned by all $\varphi^{(n)}(0)$;

(c) the closed subspace spanned by $\varphi^{(n)}(z)$ for a given z and all n does not depend on z;

(d) the closed subspace spanned by $\varphi(x)$ for all $x \in \mathbb{R}$ is equal to the closed subspace spanned by $\varphi(\mathrm{i}x)$ for all $x \in \mathbb{R}$;

(e) moreover, for an arbitrary set $A \subset \mathbb{C}$ having at least one (finite) accumulation point, the closed subspace spanned by $\varphi(z)$ for $z \in A$ does not depend on the choice of A.

If $\varphi : \mathbb{C} \to H_1$ is an entire vector-function and $U : H_1 \to H_2$ is a linear isometric embedding (of the Hilbert space H_1 to the Hilbert space H_2) then $z \mapsto U\varphi(z)$ is an entire vector-function.

1c8 Theorem. Let H_1, H_2 be Hilbert spaces and $\varphi_1 : \mathbb{C} \to H_1$, $\varphi_2 : \mathbb{C} \to H_2$ entire vector-functions. If

$$\langle \varphi_1(x), \varphi_1(y) \rangle = \langle \varphi_2(x), \varphi_2(y) \rangle \quad \text{for all } x, y \in \mathbb{R}$$

then

$$\langle \varphi_1(z_1), \varphi_1(z_2) \rangle = \langle \varphi_2(z_1), \varphi_2(z_2) \rangle \quad \text{for all } z_1, z_2 \in \mathbb{C}\,.$$

Proof (sketch). We may assume that the closed spanned subspaces are the whole spaces. We take unitary $U : H_1 \to H_2$ such that $U\varphi_1(x) = \varphi_2(x)$ for all $x \in \mathbb{R}$ (recall 'uniqueness'). It follows that $U\varphi_1(z) = \varphi_2(z)$ for all $z \in \mathbb{C}$. □

Back to the exponential map on $\mathbb{C}$

1c9 Exercise. If $f : \mathbb{C} \to \mathbb{C}$ is an entire function and $\varphi : \mathbb{C} \to H$ an entire vector-function then $f\varphi : z \mapsto f(z)\varphi(z)$ is an entire vector-function. Prove it.

1c10 Exercise. The vector-function $\varphi : \mathbb{C} \to L_2(\mathbb{R})$ defined by $\varphi(z) : q \mapsto \exp(-q^2/2 + zq)$ is entire.

Prove it.

Hint: $\Big| \mathrm{e}^{-q^2/2} \sum_{k=0}^n \frac{(zq)^k}{k!} \Big| \le \exp(-q^2/2 + |zq|)$, a majorant in L_2; use the dominated convergence theorem.

1c11 Corollary. The vector-function $\psi_1 : \mathbb{C} \to L_2(\mathbb{R})$ defined by (1c2) is entire.

By the theorem,

$$\langle \psi_1(z_1), \psi_1(z_2) \rangle = \exp(z_1 \overline{z}_2/2) \quad \text{for all } z_1, z_2 \in \mathbb{C}\,;$$

especially, $\langle \psi_1(\mathrm{i}x), \psi_1(\mathrm{i}y) \rangle = \mathrm{e}^{xy/2} = \langle \psi_1(x), \psi_1(y) \rangle$ for $x, y \in \mathbb{R}$.

Defining $w_1 : \mathbb{C} \to L_2(\mathbb{R})$ by

$$w_1(z) = \frac{\psi_1(z)}{\|\psi_1(z)\|} = \mathrm{e}^{-|z|^2/4} \psi_1(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4} - \frac{|z|^2}{4} \Big)$$

(not an analytic function) we get

$$\langle w_1(z_1), w_1(z_2) \rangle = \exp\Big(\frac12 z_1 \overline{z}_2 - \frac14 |z_1|^2 - \frac14 |z_2|^2 \Big) = \exp\Big(-\frac14 |z_1 - z_2|^2 + \frac{\mathrm{i}}{2} \mathrm{Im}\,(z_1 \overline{z}_2) \Big)\,;$$

especially, $\langle w_1(\mathrm{i}x), w_1(\mathrm{i}y) \rangle = \mathrm{e}^{-|x-y|^2/4} = \langle w_1(x), w_1(y) \rangle$ for $x, y \in \mathbb{R}$.

1c12 Lemma. The closed subspace spanned by $\{\psi_1(\mathrm{i}x) : x \in \mathbb{R}\}$ is the whole $L_2(\mathbb{R})$.

Proof (sketch). $\psi_1(\mathrm{i}x)(q) = \mathrm{const}(x) \cdot \mathrm{e}^{\mathrm{i}xq} \cdot \mathrm{e}^{-q^2/2}$; in L_2 with the weight e^{-q^2} the closed subspace spanned by the functions $q \mapsto \mathrm{e}^{\mathrm{i}xq}$ contains all periodic functions (recall Fourier series), therefore, all functions. $\square$

1c13 Corollary. The closed subspace spanned by $\{\psi_1(x) : x \in \mathbb{R}\}$ is the whole $L_2(\mathbb{R})$.

Recall the shift operators $U_1(a) = U^{(w_1)}(a)$ defined by $U_1(a)w_1(x) = w_1(x-a)$ for $a \in \mathbb{R}$ and satisfying $U_1(a)f : q \mapsto f(q+a)$ for every f of the spanned subspace. Now we see that

$$U_1(a) \in \mathrm{Unitary}(L_2(\mathbb{R}))\,, \quad U_1(a)f : q \mapsto f(q+a) \quad \text{for all } f \in L_2(\mathbb{R}),\ a \in \mathbb{R}\,.$$

Replacing $w_1(x)$ with $w_1(\mathrm{i}x)$ we get another one-parameter group of unitary operators $V_1(b)$ defined by

$$V_1(b)w_1(\mathrm{i}y) = w_1\bigl(\mathrm{i}(y+b)\bigr)$$

for $b \in \mathbb{R}$; they act on the spanned subspace, thus, on the whole $L_2(\mathbb{R})$. Taking into account that

$$\begin{gathered} w_1(\mathrm{i}y)(q) = \pi^{-1/4} \exp\Bigl(-\frac{q^2}{2} + \mathrm{i}yq\Bigr)\,, \\ w_1\bigl(\mathrm{i}(y+b)\bigr)(q) = \mathrm{e}^{\mathrm{i}bq} w_1(\mathrm{i}y)(q) \end{gathered}$$

we get

$$V_1(b) \in \mathrm{Unitary}(L_2(\mathbb{R}))\,, \quad V_1(b)f : q \mapsto \mathrm{e}^{\mathrm{i}bq} f(q) \quad \text{for all } f \in L_2(\mathbb{R}),\ b \in \mathbb{R}\,.$$

1d Fourier transform

Now we are in position to define a unitary operator $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by

$$\mathcal{F}\psi_1(x) = \psi_1(-\mathrm{i}x) \quad \text{for } x \in \mathbb{R}\,.$$

We have $\mathcal{F}U_1(a)w_1(x) = \mathcal{F}w_1(x-a) = w_1\bigl(-\mathrm{i}(x-a)\bigr) = V_1(a)w_1(-\mathrm{i}x) = V_1(a)\mathcal{F}w_1(x)$, which means that $\mathcal{F}U_1(a) = V_1(a)\mathcal{F}$, that is,

$$\mathcal{F}U_1(a)\mathcal{F}^{-1} = V_1(a) \quad \text{for } a \in \mathbb{R}\,.$$

We see that the two one-parameter unitary groups are unitarily equivalent. Shift operators are thus diagonalized!

1e Operators commuting with shifts

Every operator commuting with shifts results from shifts. Also, the Fourier transform is the only operator diagonalizing the shifts.

A linear electric circuit transforms a signal by an operator commuting with (time) shifts, provided that the circuit does not change in time. Also a linear evolution equations with constant coefficients, such as the heat equation and the Schrödinger equation, lead to evolution operators commuting with (space) shifts.

1e1 Theorem. Let a bounded linear operator $A : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ satisfy $U_1(a)A = AU_1(a)$ for all $a \in \mathbb{R}$. Then A is a strong limit of a sequence of linear combinations of operators $U_1(a)$.

That is, there exist $c_k^{(n)} \in \mathbb{C}$ and $a_k^{(n)} \in \mathbb{R}$ such that

$$\forall f \in L_2(\mathbb{R}) \quad \Big\| \sum_{k=1}^n c_k^{(n)} U_1\big(a_k^{(n)}\big) f - Af \Big\| \to 0 \quad \text{as } n \to \infty \,.$$

1e2 Remark. (a) The theorem fails for norm convergence of operators (as we'll see later). (b) The theorem fails for $L_2(\mathbb{R} \to \mathbb{C}^2)$ (as we can see immediately).

Proof (sketch). The proof consists of four steps.

First step: we may replace $U_1(\cdot)$ with $V_1(\cdot)$. *Proof:* unitary equivalence.

Second step: A is a multiplication operator, that is,

$$\exists \varphi \in L_\infty(\mathbb{R}) \;\; \forall f \in L_2(\mathbb{R}) \;\; Af = \varphi \cdot f \,.$$

Proof: define $f_0(t) = \mathrm{e}^{-t^2/2}$, $g_0 = Af_0$ and $\varphi = g_0/f_0$, then $AV_1(b)f_0 = V_1(b)g_0 = \varphi \cdot V_1(b)f_0$ for all b. Linear combinations of functions $V_1(b)f_0$ are dense in L_2; it follows that $\varphi \in L_\infty$ (since A is bounded) and $Af = \varphi \cdot f$ for all $f \in L_2$.

Third step: it is sufficient to find trigonometric polynomials p_n such that $p_n \to \varphi$ almost everywhere and $\sup_{n,q} |p_n(q)| < \infty$. *Proof:* then for every $f \in L_2$ we have $\|p_n \cdot f - \varphi \cdot f\| \to 0$ by the dominated convergence theorem, since $|p_n \cdot f - \varphi \cdot f|^2 \le \text{const} \cdot |f|^2$.

Last step. We know that trigonometric polynomials are dense in L_2 with weight e^{-q^2}. Moreover, having $\varphi \in L_\infty(\mathbb{R})$ we get (by the same Fourier series argument, via Feier sums) trigonometric polynomials p_n such that $\|p_n\|_\infty \le \|\varphi\|_\infty$ and $\int |p_n(q) - \varphi(q)|^2 \mathrm{e}^{-q^2} \,\mathrm{d}q \to 0$ as $n \to \infty$. Taking a subsequence we get $\sum_n \int |p_n(q) - \varphi(q)|^2 \mathrm{e}^{-q^2} \,\mathrm{d}q < \infty$, which implies $\sum_n |p_n(q) - \varphi(q)|^2 < \infty$ almost everywhere. $\square$

Now we can see that the theorem fails for norm convergence of operators; it happens because trigonometric polynomials are not dense in $L_\infty(\mathbb{R})$.

As a by-product we get the general form of an operator commuting with all $V_1(b)$, — just the multiplication operator

$$L_2(\mathbb{R}) \ni f \mapsto \varphi \cdot f\,, \quad \varphi \in L_\infty(\mathbb{R})\,,$$

and, more important, the general form of an operator commuting with all $U_1(a)$,

$$L_2(\mathbb{R}) \ni f \mapsto \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)\,, \quad \varphi \in L_\infty(\mathbb{R})\,.$$

1e3 Corollary. If an operator commutes with all $U_1(a)$ and all $V_1(b)$ then it is scalar.

Proof: it is multiplication by φ, and φ is shift-invariant.

1e4 Corollary. If an invertible operator $\mathcal{F}_1$ satisfies $\mathcal{F}_1 U_1(a) \mathcal{F}_1^{-1} = V_1(a)$ for all $a \in \mathbb{R}$ then $\mathcal{F}_1 = c\mathcal{F}$ for some $c \in \mathbb{C}$.

Proof: $\mathcal{F}_1^{-1}\mathcal{F}$ commutes with all $U_1(a)$ and $V_1(b)$.

The same holds for $U(a)$ and $V(b)$ in general.

1f Inverse Fourier transform

The relation $\mathcal{F}\psi_1(z) = \psi_1(-\mathrm{i}z)$ holds for $z \in \mathbb{R}$, therefore, for all $z \in \mathbb{C}$. In particular, $\mathcal{F}\psi_1(\mathrm{i}x) = \psi_1(x)$ for $x \in \mathbb{R}$. We note that $\langle \psi_1(z_1), \psi_1(z_2) \rangle = \exp(z_1 \overline{z}_2/2) = \langle \psi_1(-z_1), \psi_1(-z_2) \rangle$, introduce a unitary operator $J : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by

$$J\psi_1(z) = \psi_1(-z)$$

and get $\mathcal{F}J\psi_1(-\mathrm{i}x) = \psi_1(x)$, that is, $\mathcal{F}J\mathcal{F}\psi_1(x) = \psi_1(x)$, which means that $\mathcal{F}J\mathcal{F} = 1\!\!1$ and so,

$$\mathcal{F}^{-1} = \mathcal{F}J = J\mathcal{F}\,.$$

It remains to note that $J\psi_1(z)(q) = \psi_1(-z)(q) = \pi^{-1/4} \exp\bigl(-\frac{q^2}{2} - zq - \frac{z^2}{4}\bigr) = \psi_1(z)(-q)$, thus,

$$(Jf)(q) = f(-q) \quad \text{for } f \in L_2(\mathbb{R})\,.$$

1g Convolution operators

Integral combinations of shift operators are convolution operators.

We want to define an operator $\int g(a) U_1(-a)\,\mathrm{d}a$ for every $g \in L_1(\mathbb{R})$.[1] One approach is, to integrate the vector-function $a \mapsto U_1(-a)f$ (for a given

[1]The minus sign is traditional.

$f \in L_2(\mathbb{R})$) with the weight g. Another approach is the classical convolution formula $(f * g)(q) = \int g(a) f(q-a)\,\mathrm{d}a$.

Integrating the vector-function

1g1 Lemma. The one-parameter unitary group $U_1(\cdot)$ is strongly continuous.

That is, $\|f - U_1(a)f\| \to 0$ as $a \to 0$ for every $f \in L_2(\mathbb{R})$. Continuity at other points follows easily from continuity at 0.

First proof. Using the unitary equivalence we replace $U_1(\cdot)$ with $V_1(\cdot)$; now, $\int |f(q) - \mathrm{e}^{\mathrm{i}bq} f(q)|^2\,\mathrm{d}q \to 0$ as $b \to 0$ by the dominated convergence theorem. $\square$

1g2 Remark. However, $\|\mathbb{1} - U_1(a)\| = 2$ for all $a \ne 0$, since $\operatorname{ess\,sup}_{t\in\mathbb{R}} |1 - \mathrm{e}^{\mathrm{i}bt}| = 2$ for all $b \ne 0$.

Another proof. The claim holds evidently on a dense set of f, which is sufficient. $\square$

Given $f \in L_2(\mathbb{R})$, we have a bounded continuous vector-function $\mathbb{R} \ni a \mapsto U_1(-a)f$, and may consider the improper Riemann-Stieltjes integral

$$\int_{-\infty}^{+\infty} U_1(-a) f \,\mathrm{d}G(a) = \lim_{C\to+\infty} \lim_{n\to\infty} \sum_{k=-n}^{n-1} \Big(G\Big(\frac{k+1}{n}C\Big) - G\Big(\frac{k}{n}C\Big)\Big) U_1\Big(-\frac{k}{n}C\Big) f\,,$$

where $G(a) = \int_{-\infty}^a g(q)\,\mathrm{d}q$; norm convergence in $L_2(\mathbb{R})$ is proved in the same way as for scalar-valued functions (and of course, it holds for arbitrary partitions and points). This integral will be denoted by $\int g(a) U_1(-a) f\,\mathrm{d}a$. Clearly, $\| \int g(a) U_1(-a) f\,\mathrm{d}a\|_2 \le \|g\|_1 \|f\|_2$, and we define a bounded linear operator $\int g(a) U_1(-a)\,\mathrm{d}a$ as $f \mapsto \int g(a) U_1(-a) f\,\mathrm{d}a$. Thus, $\| \int g(a) U_1(-a)\,\mathrm{d}a\| \le \|g\|_1$; we have a bounded linear map from $L_1(\mathbb{R})$ to the space of operators $L_2(\mathbb{R}) \to L_2(\mathbb{R})$. Clearly, $\int g(a) U_1(-a)\,\mathrm{d}a$ commutes with shifts.

It is easy to guess that

$$\int g(a) U_1(-a) f\,\mathrm{d}a : q \mapsto \int g(a) f(q-a)\,\mathrm{d}a\,,$$

but not so easy to prove it, since $f(t)$ cannot be treated as a continuous linear functional of $f \in L_2$.

Classical convolution formula

1g3 Lemma. Let $f \in L_2(\mathbb{R})$ and $g \in L_1(\mathbb{R})$, then for almost all $q \in \mathbb{R}$,

(a) $\int |f(q-a)g(a)| \, \mathrm{d}a < \infty$,

(b) $\int g(a)U_1(-a)f \, \mathrm{d}a : q \mapsto \int g(a)f(q-a) \, \mathrm{d}a$.

Proof. (a) $\int \mathrm{d}q \int \mathrm{d}a \, |f(q-a)|^2 |g(a)| = \int \mathrm{d}a \, |g(a)| \int \mathrm{d}q \, |f(q-a)|^2 = \|g\|_1 \|f\|_2^2 < \infty$, therefore $\int |f(q-a)|^2 |g(a)| \, \mathrm{d}a < \infty$ for almost all q, which implies $\int |f(q-a)g(a)| \, \mathrm{d}a \le \big(\int |f(q-a)|^2 |g(a)| \, \mathrm{d}a \big)^{1/2} \big(\int |g(a)| \, \mathrm{d}a \big)^{1/2} < \infty$.

(b) The function $q \mapsto \int f(q-a)g(a) \, \mathrm{d}a$ belongs to L_2, since $\int \mathrm{d}q \big| \int f(q-a)g(a) \, \mathrm{d}a \big|^2 \le \|g\|_1 \cdot \int \mathrm{d}q \int \mathrm{d}a \, |f(q-a)|^2 |g(a)| \le \|g\|_1^2 \|f\|_2^2$. It is sufficient to prove the equality

$$\Big\langle \int g(a)U_1(-a)f \, \mathrm{d}a, h \Big\rangle = \Big\langle q \mapsto \int f(q-a)g(a) \, \mathrm{d}a, h \Big\rangle$$

for all $h \in L_2(\mathbb{R})$. The linear functional $\langle \cdot, h \rangle$ applies to the Riemann-Stieltjes integral, giving

$$\Big\langle \int g(a)U_1(-a)f \, \mathrm{d}a, h \Big\rangle = \int g(a) \langle U_1(-a)f, h \rangle \, \mathrm{d}a = \int \mathrm{d}a \, g(a) \int \mathrm{d}q \, f(q-a)h(t) \, .$$

The needed equality

$$\int \mathrm{d}a \, g(a) \int \mathrm{d}q \, h(q)f(q-a) = \int \mathrm{d}q \, h(q) \int \mathrm{d}a \, g(a)f(q-a)$$

follows from Fubini's theorem, since

$$\begin{aligned} \iint \mathrm{d}a \mathrm{d}q \, |g(a)h(q)f(q-a)| &= \\ &= \int \mathrm{d}q \, |h(q)| \int \mathrm{d}a \, |g(a)f(q-a)| \le \\ &\le \|h\|_2^{1/2} \cdot \Big\| q \mapsto \int \mathrm{d}a \, |g(a)f(q-a)| \Big\|_2^{1/2} < \infty \, . \end{aligned}$$

$\square$

Thus we have two equivalent definitions of the convolution $f * g$ for $f \in L_2(\mathbb{R})$, $g \in L_1(\mathbb{R})$,

$$f * g = \int g(a)U_1(-a)f \, \mathrm{d}a \, ;$$

$$f * g : t \mapsto \int f(t-s)g(s) \, \mathrm{d}s \, .$$

The latter formula shows that

$$f * g = g * f \quad \text{for } f, g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \, .$$

1h Fourier transform and convolution

The convolution operator $\int g(a)U_1(-a)\,\mathrm{d}a : f \mapsto f * g$ commutes with all $U_1(a)$, therefore it must be $f \mapsto \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)$ for some $\varphi \in L_\infty(\mathbb{R})$. We have

$$\mathcal{F}\sum_k c_k U_1(-a_k)\mathcal{F}^{-1} = \sum_k c_k V_1(-a_k)\,;$$

the limiting procedure (strong convergence...) gives

$$\mathcal{F}\biggl(\int g(a)U_1(-a)\,\mathrm{d}a\biggr)\mathcal{F}^{-1} = \int g(a)V_1(-a)\,\mathrm{d}a\,,$$

which means that

$$\varphi(q) = \int g(a)\mathrm{e}^{-\mathrm{i}aq}\,\mathrm{d}a\,;$$

$$f * g = \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)\,, \quad \text{that is,} \quad \mathcal{F}(f * g) = \varphi \cdot \mathcal{F}f$$

for $f \in L_2(\mathbb{R})$ and $g \in L_1(\mathbb{R})$.

1i Gaussian functions

We know that $\mathcal{F}\psi_1(0) = \psi_1(0)$ and $\psi_1(0) : q \mapsto \pi^{-1/4}\mathrm{e}^{-q^2/2}$. Now we rescale that function: $f_\sigma \in L_2(\mathbb{R})$,

$$f_\sigma(q) = \pi^{-1/4}\sigma^{-1/2}\exp\Bigl(-\frac{q^2}{2\sigma^2}\Bigr) \quad \text{for } 0 < \sigma < \infty\,;$$

note that $f_1 = \psi_1(0)$, $\mathcal{F}f_1 = f_1$. The functions f_σ are normalized in L_2: $\|f_\sigma\| = 1$. Also Gaussian functions normalized in L_1 are useful,

$$g_\sigma = \frac{f_\sigma}{\|f_\sigma\|_1} = 2^{-1/2}\pi^{-1/4}\sigma^{-1/2}f_\sigma : q \mapsto (2\pi)^{-1/2}\sigma^{-1}\exp\Bigl(-\frac{q^2}{2\sigma^2}\Bigr)\,.$$

1i1 Lemma. $\mathcal{F}f_\sigma = f_{1/\sigma}$ for $0 < \sigma < \infty$.

Proof. It is sufficient to prove it for $\sigma \ge 1$, since it implies $f_\sigma = \mathcal{F}^{-1}f_{1/\sigma} = \mathcal{F}Jf_{1/\sigma} = \mathcal{F}f_{1/\sigma}$.

Given $\sigma > 1$ we introduce $\delta = \sqrt{\sigma^2 - 1}$ and note that $f_1 * g_\delta = \sigma^{-1/2}f_\sigma$ (check it). On the other hand, $\mathcal{F}(f_1 * g_\delta) = \varphi_\delta \cdot \mathcal{F}f_1$ where $\varphi_\delta(q) = \int g_\delta(a)\mathrm{e}^{-\mathrm{i}aq}\,\mathrm{d}a = \exp\bigl(-\frac{1}{2}\delta^2 q^2\bigr)$ (check it). Thus, $\mathcal{F}f_\sigma = \sigma^{1/2}\varphi_\delta \cdot f_1 = f_{1/\sigma}$ (check it). $\square$

1i2 Corollary. $\mathcal{F}g_\sigma = \frac{1}{\sigma}g_{1/\sigma}$.

1j Explicit formula

1j1 Theorem.

$$\mathcal{F}f : t \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}st} f(s)\,\mathrm{d}s \quad \text{for } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,.$$

We use the Gaussian functions f_σ, g_σ.

1j2 Lemma. $f * g_\sigma \to f$ in L_2 as $\sigma \to 0+$, for every $f \in L_2$.

Proof. $\|f * g_\sigma - f\| = \|\int g_\sigma(a)(U_1(-a)f - f)\,\mathrm{d}a\| \le \int g_\sigma(a)\|U_1(-a)f - f\|\,\mathrm{d}a = \int_{(-\varepsilon,+\varepsilon)} \dots + \int_{\mathbb{R}\setminus(-\varepsilon,+\varepsilon)} \dots \to 0.$ □

Proof of the theorem. $\mathcal{F}(f * g_\sigma) = \mathcal{F}(g_\sigma * f) = \varphi \cdot \mathcal{F}g_\sigma$, where $\varphi : t \mapsto \int f(s)\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s$. We have $\mathcal{F}g_\sigma = \frac{1}{\sigma}g_{1/\sigma} : t \mapsto (2\pi)^{-1/2}\mathrm{e}^{-\sigma^2t^2/2} \to (2\pi)^{-1/2}$ as $\sigma \to 0+$ pointwise (moreover, locally uniformly). On the other hand, $\varphi \cdot \frac{1}{\sigma}g_{1/\sigma} = \varphi \cdot \mathcal{F}g_\sigma = \mathcal{F}(f * g_\sigma) \to \mathcal{F}(f)$ in L_2. It follows that $\varphi \in L_2$ and $\mathcal{F}(f) = (2\pi)^{-1/2}\varphi$. □

1j3 Corollary. $\mathcal{F}(f * g) = (2\pi)^{1/2}(\mathcal{F}f) \cdot (\mathcal{F}g)$ for $f \in L_2$, $g \in L_1 \cap L_2$.

1k List of results

1k1 Plancherel's theorem

There exists a bounded linear operator (called Fourier transform) $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ such that

$$\mathcal{F}f : t \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}st} f(s)\,\mathrm{d}s \quad \text{for all } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,.$$

The operator $\mathcal{F}$ is isometric, which means the following.

Plancherel's formula:

$$\|\mathcal{F}f\| = \|f\| \quad \text{for all } f \in L_2(\mathbb{R})\,.$$

Parseval's formula:

$$\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, g \rangle \quad \text{for all } f, g \in L_2(\mathbb{R})\,.$$

1k2 Inversion formula

The isometric operator $\mathcal{F}$ is unitary, which means that $\mathcal{F}\big(L_2(\mathbb{R})\big) = L_2(\mathbb{R})$. The inverse operator $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1} = \mathcal{F}J = J\mathcal{F}$$

where J is defined by $Jf : t \mapsto f(-t)$.

1k3 Diagonalization of shifts

First,

$$\mathcal{F}U(a)\mathcal{F}^{-1} = V(a) \quad \text{for all } a \in \mathbb{R}\,,$$

where operators $U(a), V(a) : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ are defined by

$$U(a)f : t \mapsto f(t+a)\,, \quad V(a)f : t \mapsto \mathrm{e}^{\mathrm{i}at} f(t)\,.$$

Second, this property characterizes $\mathcal{F}$ uniquely up to a coefficient.

1k4 Diagonalization of the Fourier transform

There exists an orthonormal basis $(e_0, e_1, e_2, \dots)$ of $L_2(\mathbb{R})$ such that

$$\mathcal{F}e_k = (-\mathrm{i})^k e_k \quad \text{for } k = 0, 1, 2, \dots$$

Namely,

$$e_k = \frac{2^{k/2}}{\sqrt{k!}} \frac{\mathrm{d}^k}{\mathrm{d}z^k}\bigg|_{z=0} \psi(z)\,,$$

where $\psi : \mathbb{C} \to L_2(\mathbb{R})$ is defined by

$$\psi(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4} \Big)\,.$$

1k5 Some complete systems in $L_2(\mathbb{R})$

The vectors e_k mentioned above are a complete system in the sense that their linear combinations are dense in $L_2(\mathbb{R})$. Also functions

$$q \mapsto \exp\big(-(q-x)^2 \big)$$

for all $x \in \mathbb{R}$ are a complete system.

1k6 Fourier transform and convolution

$$\mathcal{F}(f * g) = (2\pi)^{1/2} (\mathcal{F}f) \cdot (\mathcal{F}g) \quad \text{for all } f \in L_2(\mathbb{R}),\ g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,;$$

here

$$f * g : t \mapsto \int f(t-s) g(s)\,\mathrm{d}s\,.$$

1k7 Operators commuting with shifts

A bounded linear operator on $L_2(\mathbb{R})$ commutes with all $U(a)$ if and only if it is of the form

$$f \mapsto \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)$$

for some $\varphi \in L_\infty(\mathbb{R})$.

1k8 Vector-functions

In addition we have several statements about vector-functions (both specific and general) valued in a (general) Hilbert space, their derivatives and integrals. See especially Sections 1c and 1g.

1l List of formulas

$x, y \in \mathbb{R}$; $z_1, z_2 \in \mathbb{C}$; $\psi_0, w_0 : \mathbb{C} \to H$;

$$\langle \psi_0(z_1), \psi_0(z_2) \rangle = \exp\Big(\frac12 z_1 \overline{z}_2 \Big) \,; \tag{1l1}$$

$$w_0(z) = \frac{\psi_0(z)}{\|\psi_0(z)\|} = \mathrm{e}^{-|z|^2/4} \psi_0(z) \,; \tag{1l2}$$

$$\langle w_0(z_1), w_0(z_2) \rangle = \exp\Big(-\frac14 |z_1 - z_2|^2 + \frac{\mathrm{i}}{2} \operatorname{Im}(z_1 \overline{z}_2) \Big) \,; \tag{1l3}$$

$$\langle w_0(x), w_0(y) \rangle = \mathrm{e}^{-|x-y|^2/4} = \langle w_0(\mathrm{i}x), w_0(\mathrm{i}y) \rangle \,; \tag{1l4}$$

$U_0, V_0 : \mathbb{R} \to \text{Unitary}(H)$;

$$U_0(a) w_0(x) = w_0(x - a) \,; \tag{1l5}$$

$$V_0(b) w_0(\mathrm{i}y) = w_0\big(\mathrm{i}(y + b)\big) \,; \tag{1l6}$$

$\psi_1, w_1 : \mathbb{C} \to L_2(\mathbb{R})$;

$$\psi_1(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4} \Big) \,; \tag{1l7}$$

$$w_1(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4} - \frac{|z|^2}{4} \Big) \,; \tag{1l8}$$

$$w_1(x) : q \mapsto \pi^{-1/4} \exp\Big(-\frac12 (q - x)^2 \Big) \,; \tag{1l9}$$

$$w_1(\mathrm{i}y) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + \mathrm{i}yq \Big) \,; \tag{1l10}$$

$U_1, V_1 : \mathbb{R} \to \text{Unitary}(L_2(\mathbb{R}))$, $a, b \in \mathbb{R}$;

$$U_1(a) f : q \mapsto f(q + a) \,; \tag{1l11}$$

$$V_1(b) f : q \mapsto \mathrm{e}^{\mathrm{i}bq} f(q) \,; \tag{1l12}$$

$\mathcal{F} \in \mathrm{Unitary}(L_2(\mathbb{R}))$;

(1l13) $$\mathcal{F}\psi_1(z) = \psi_1(-\mathrm{i}z)\,;$$

(1l14) $$\mathcal{F}U_1(a)\mathcal{F}^{-1} = V_1(a)\,;$$

(1l15) $$\mathcal{F}^{-1} = \mathcal{F}J = J\mathcal{F}\,, \quad Jf : q \mapsto f(-q)\,;$$

(1l16) $$\mathcal{F}f : t \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}st} f(s)\,\mathrm{d}s \quad \text{for } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,;$$

(1l17) $$f * g = \int g(a)U_1(-a)f\,\mathrm{d}a : t \mapsto \int f(t-s)g(s)\,\mathrm{d}s\,;$$

(1l18) $$f * g = g * f \quad \text{for } f, g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,;$$

(1l19) $$\mathcal{F}(f * g) = (2\pi)^{1/2}(\mathcal{F}f)\cdot(\mathcal{F}g) \quad \text{for } f \in L_2(\mathbb{R}),\ g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,.$$

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