2 Functions of the differentiation operator

This preparatory chapter aims at some acquaintance with unbounded operators and functions of them. Postponing the general theory, here we treat functions of the differentiation operator on $L_2(\mathbb{R})$ using the Fourier transform.

2a Introduction

For a diagonal matrix $A = \operatorname{diag}(a_1, \dots, a_n)$ we have $p(A) = \operatorname{diag}\big(p(a_1), \dots, p(a_n)\big)$ for every polynomial p. For a diagonalizable matrix A we have $FAF^{-1} = \operatorname{diag}(a_1, \dots, a_n)$ for some (invertible) matrix F, and $Fp(A)F^{-1} = p(FAF^{-1}) = \operatorname{diag}\big(p(a_1), \dots, p(a_n)\big)$. It is natural to define

$$\varphi(A) = F^{-1} \operatorname{diag}\big(\varphi(a_1), \dots, \varphi(a_n)\big) F$$

for every $\varphi : \{a_1, \dots, a_n\} \to \mathbb{C}$. The result does not depend on the choice of F. The map $\varphi \mapsto \varphi(A)$ is a homomorphism of algebras, that is,

linearity: $(a\varphi + b\psi)(A) = a\varphi(A) + b\psi(A)$,

multiplicativity: $(\varphi \cdot \psi)(A) = \varphi(A)\psi(A)$

for all $a, b \in \mathbb{C}$ and $\varphi, \psi \in \mathbb{C}^{\{a_1,\dots,a_n\}}$. Note also

unit preservation: $\mathbb{1}(A) = \mathbb{1}$.

Assume in addition that $A^* = A$, then $a_1, \dots, a_n \in \mathbb{R}$, F can be chosen unitary, and the homomorphism is a $*$-homomorphism, that is,

involution preservation: $\overline{\varphi}(A) = (\varphi(A))^*$

for all $\varphi \in \mathbb{C}^{\{a_1,\dots,a_n\}}$. In particular, $\varphi(A)$ is self-adjoint for all $\varphi \in \mathbb{R}^{\{a_1,\dots,a_n\}}$. Note also

positivity: if $\varphi \ge 0$ then $\varphi(A) \ge 0$

for all $\varphi \in \mathbb{R}^{\{a_1,\dots,a_n\}}$.

For a compact self-adjoint operator in a Hilbert space the situation is similar; a finite spectrum $\{a_1, \dots, a_n\}$ is replaced with a sequence converging to 0.

For a bounded (not just compact) self-adjoint operator in a Hilbert space the situation is similar in principle, but more complicated technically, because of (possibly) continuous spectrum. Additional technical complications appear for unbounded self-adjoint operators.

In this chapter we consider mostly the (unbounded) differentiation operator in $L_2(\mathbb{R})$, which is rather easy due to its diagonalization by the Fourier transform.

2b Multiplication operators

All multiplication operators are functions of one important operator Q, the generator of the unitary group $\big(V(b)\big)_{b\in\mathbb{R}}$.

We know that $L_\infty(\mathbb{R})$ acts on $L_2(\mathbb{R})$ by multiplication operators,

$$L_2 \ni f \mapsto \varphi \cdot f \in L_2 \,, \qquad \varphi \in L_\infty \,.$$

2b1 Exercise. Formulate and prove the five properties of this action:
linearity,
multiplicativity,
unit preservation,
involution preservation,
positivity.

What about multiplication

$$f \mapsto \big(q \mapsto qf(q)\big)$$

by the unbounded function $q \mapsto q$? Surely it is not a bounded operator. We define

$$D_Q = \left\{ f \in L_2(\mathbb{R}) : \int q^2 |f(q)|^2 \, \mathrm{d}q < \infty \right\} ,$$
$$Q : D_Q \to H \,,$$
$$Qf : q \mapsto qf(q) \quad \text{for } f \in D_Q \,;$$

Q is an example of so-called "densely defined unbounded linear operator", and the dense linear set D_Q is its domain. Similarly, for every $\varphi \in L_0(\mathbb{R})$ (just a measurable function $\mathbb{R} \to \mathbb{C}$) we define

$$\begin{gathered} D_\varphi = \{ f \in L_2(\mathbb{R}) : \varphi \cdot f \in L_2(\mathbb{R}) \} \,, \\ A_\varphi : D_\varphi \to H \,, \\ A_\varphi f = \varphi \cdot f \quad \text{for } f \in D_\varphi \,; \end{gathered}$$

A_φ is a densely defined linear operator, unbounded unless $\varphi \in L_\infty$. The special case $\varphi = \mathrm{id} : q \mapsto q$ leads to the operator $A_{\mathrm{id}} = Q$.

2b2 Exercise. If $\varphi, \psi \in L_0$ satisfy $\varphi - \psi \in L_\infty$ then

$$\begin{gathered} D_\varphi = D_\psi \,, \\ A_\varphi f - A_\psi f = (\varphi - \psi) \cdot f \quad \text{for } f \in D_\varphi = D_\psi \,. \end{gathered}$$

Prove it.

In particular, $D_{\mathrm{id}+c\mathbf{1}} = D_{\mathrm{id}} = D_Q$ for each $c \in \mathbb{C}$, and $A_{\mathrm{id}+c\mathbf{1}} = Q + c1\!\!1$.

2b3 Exercise. Let $\varphi \in L_\infty$, $\psi \in L_0$, then

$$\begin{gathered} D_{\varphi\cdot\psi} = \{ f : \varphi \cdot f \in D_\psi \} \supset D_\psi \,, \\ (\varphi \cdot \psi) \cdot f = \psi \cdot (\varphi \cdot f) \quad \text{for } f \in D_{\varphi\cdot\psi} \,. \end{gathered}$$

The relations $D_{\varphi\cdot\psi} = D_\psi$ and $(\varphi \cdot \psi) \cdot f = \varphi \cdot (\psi \cdot f)$ (for $f \in D_{\varphi\cdot\psi}$) are generally wrong; however, they hold if $|\varphi(\cdot)|$ is bounded away from 0.

Prove the positive claims, and find counterexamples to the negative claims.

An interesting special case is well-known as Cayley transform. Given $\psi \in L_0$ such that $\psi = \overline{\psi}$, we introdice $\varphi \in L_\infty$ by

$$\varphi(x) = \frac{\psi(x) - \mathrm{i}}{\psi(x) + \mathrm{i}} \,,$$

observe that $|\varphi(\cdot)| = 1$ and $\psi - \mathrm{i}1\!\!1 = \varphi \cdot (\psi + \mathrm{i}1\!\!1)$, therefore A_φ is unitary and $(\psi - \mathrm{i}1\!\!1) \cdot f = (\psi + \mathrm{i}1\!\!1) \cdot (\varphi \cdot f)$, which leads to a remarkable relation between the unbounded[1] self-adjoint operator $A = A_\psi$ and the unitary operator $U = A_\varphi$:

(2b4) $$(A - \mathrm{i}1\!\!1) f = (A + \mathrm{i}1\!\!1) U f \quad \text{for } f \in D_A$$

[1] Here and henceforth I often write "unbounded" meaning "generally, unbounded", that is, "not necessarily bounded".

(which determines U uniquely), and

$$(\mathbb{1} - U)Af = \mathrm{i}(\mathbb{1} + U)f \quad \text{for } f \in D_A$$

(since $(1 - \varphi) \cdot \psi = \mathrm{i}(1 + \varphi)$), which restores A from U.

Postponing the general definition of a function of operator, for now we define

$$\varphi(Q) = A_\varphi \quad \text{for } \varphi \in L_0(\mathbb{R})\,.$$

In particular, $\varphi = \mathrm{id} + c\mathbb{1} : q \mapsto q + c$ gives $\varphi(Q) = Q + c\mathbb{1}$; $\varphi : q \mapsto q^n$ gives $\varphi(Q) = Q^n$; also, $\varphi : q \mapsto \mathrm{e}^{\mathrm{i}bq}$ gives $\varphi(Q) = \exp(\mathrm{i}bQ)$.

2b5 Exercise. Let $n \in \{2, 3, \dots\}$.

(a) $Q^n f$ is defined if and only if $Q^{n-1} f$ is defined and belongs to D_Q;

(b) in this case $Q^n f = Q(Q^{n-1} f)$.

Prove it.

2b6 Exercise. (a) $Q^{-1} f$ is defined if and only if there exists $g \in D_Q$ such that $Qg = f$;

(b) in this case such g is unique, and $Q^{-1} f = g$.

Prove it.

Recall the unitary operators $V(b)$ of (1b12) (denoted there by $V_1(b)$). Clearly,

$$\exp(\mathrm{i}bQ) = V(b) \quad \text{for all } b \in \mathbb{R}\,.$$

The operator Q is the *generator* of the one-parameter unitary group $\big(V(b)\big)_{b\in\mathbb{R}}$ in the following sense.

2b7 Exercise. (a) The following three conditions are equivalent for every $f \in L_2(\mathbb{R})$:

(a1) $$\| f - \exp(\mathrm{i}\lambda Q) f \| = O(\lambda) \quad \text{as } \lambda \to 0\,;$$

(a2) $$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \exp(\mathrm{i}\lambda Q) f \quad \text{exists (in the norm);}$$

(a3) $$f \in D_Q\,.$$

(b) In this case

$$Qf = -\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \exp(\mathrm{i}\lambda Q) f\,.$$

Prove it.

Hint: $|1 - \mathrm{e}^{\mathrm{i}\lambda q}| \le |\lambda q|$; use Fatou's lemma for (a1)$\Longrightarrow$(a3), and the dominated convergence theorem for (a3)$\Longrightarrow$(a2).

2c Functions of the differentiation operator

All operators commuting with shifts are functions of one important operator P, the generator of the unitary group $\big(U(a)\big)_{a\in\mathbb{R}}$ of shifts.

Recalling the general form of an operator commuting with shifts,

$$B_\varphi f = \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)\,,$$

we observe another action $\varphi \mapsto B_\varphi$ of $L_\infty(\mathbb{R})$ on $L_2(\mathbb{R})$.

2c1 Exercise. Formulate and prove the five properties of this action:
linearity,
multiplicativity,
unit preservation,
involution preservation,
positivity.
Hint: use 2b1 and unitarity of $\mathcal{F}$.

We do the same for unbounded operators. Namely, for every $\varphi \in L_0(\mathbb{R})$ we define

$$\begin{gathered} D_{B_\varphi} = \{ f \in L_2(\mathbb{R}) : \mathcal{F}f \in D_{A_\varphi} \} = \mathcal{F}^{-1} D_{A_\varphi}\,, \\ B_\varphi : D_{B_\varphi} \to H\,, \\ B_\varphi f = \mathcal{F}^{-1}(A_\varphi \mathcal{F} f) \quad \text{for } f \in D_{B_\varphi}\,; \end{gathered}$$

B_φ is a densely defined linear operator (unbounded unless $\varphi \in L_\infty$) unitarily equivalent to $\varphi(Q)$,

$$B_\varphi = \mathcal{F}^{-1} \varphi(Q) \mathcal{F}\,,$$

and we treat it as a function of the operator $P = B_{\mathrm{id}}$:

$$\begin{gathered} P = \mathcal{F}^{-1} Q \mathcal{F}\,, \\ \varphi(P) = \mathcal{F}^{-1} \varphi(Q) \mathcal{F}\,. \end{gathered}$$

Recall the unitary operators $U(a)$ of (1b11) (denoted there by $U_1(a)$). We have

$$U(a) = \exp(\mathrm{i}aP) \quad \text{for all } a \in \mathbb{R}\,.$$

The operator P is the generator of the one-parameter unitary group $\big(U(a)\big)_{a\in\mathbb{R}}$ in the following sense.

2c2 Exercise. (a) The following three conditions are equivalent for every $f \in L_2(\mathbb{R})$:

(a1) $$\|f - U(a)f\| = O(a) \quad \text{as } a \to 0\,;$$

(a2) $$\frac{\mathrm{d}}{\mathrm{d}a}\Big|_{a=0} U(a)f \quad \text{exists (in the norm);}$$

(a3) $$f \in D_P\,.$$

(b) In this case

$$Pf = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}a}\Big|_{a=0} U(a)f\,.$$

Prove it.

Hint: use 2b7, unitarity of $\mathcal{F}$, and the equality $U(a) = \exp(\mathrm{i}aP)$.

If f is nice enough, say, continuously differentiable and compactly supported, then clearly $f' \in L_2$ and

$$U(a)f = f + af' + o(a) \quad \text{in the norm, as } a \to 0$$

(since $U(a)f : q \mapsto f(q+a)$), thus $f \in D_P$ and

$$Pf = -\mathrm{i}f'\,.$$

We see that in some sense $\mathrm{i}P$ is the differentiation operator $f \mapsto f'$. However, what happens for not so nice functions?

2c3 Theorem. The following three conditions on $f, g \in L_2(\mathbb{R})$ are equivalent:

(a) $f \in D_P$ and $\mathrm{i}Pf = g$;

(b) there exist continuously differentiable compactly supported functions $f_1, f_2, \dots$ such that

$$f_n \to f \quad \text{in } L_2\,,$$
$$f'_n \to g \quad \text{in } L_2\,;$$

(c) for every $a \in \mathbb{R}$,

$$U(a)f = f + \int_0^a U(b)g\,\mathrm{d}b\,.$$

(The latter is the Riemann integral of a continuous vector-function, recall 1g, especially 1g1.)

Proof (sketch). (a)$\Longrightarrow$(b): we take $f_n = (f \cdot 1\!\!1_{(-n,n)}) * h_n$ where h_n are "triangles" $q \mapsto \max(0, n - n^2|q|)$; then $f_n \to f$ in L_2, and $U(a)f_n = (U(a)f) * h_n$, thus $f_n' = \frac{\mathrm{d}}{\mathrm{d}a}\big|_{a=0} U(a)f_n = \big(\frac{\mathrm{d}}{\mathrm{d}a}\big|_{a=0} U(a)f\big) * h_n = g * h_n \to g$ in L_2.

(b)$\Longrightarrow$(c): $\frac{\mathrm{d}}{\mathrm{d}a} U(a)f_n = U(a)f_n'$, thus $U(a)f_n = f_n + \int_0^a U(b)f_n' \,\mathrm{d}b$; we take the limit as $n \to \infty$.

(c)$\Longrightarrow$(a): $\frac{\mathrm{d}}{\mathrm{d}a}\big|_{a=0} U(a)f = \frac{\mathrm{d}}{\mathrm{d}a}\big|_{a=0} \int_0^a U(b)g \,\mathrm{d}b = g$. □

According to 2c3(b), $f_n \to f$ in the so-called Sobolev space $W_2^1(\mathbb{R})$, and so, $D_P = W_2^1(\mathbb{R})$. Two more equivalent condition (without proof):

(d) $\langle f, h' \rangle = -\langle g, h \rangle$ for all continuously differentiable compactly supported functions h;

(e) $f(x) = \lim_{a \to -\infty} \int_a^x g(y) \,\mathrm{d}y$ for almost all x.

So, the Fourier transform diagonalizes also the differentiation operator: if $f' = g$ in the generalized sense described above (namely, $\mathrm{i}Pf = g$) then $\mathrm{i}p(\mathcal{F}f)(p) = (\mathcal{F}g)(p)$ for almost all p (namely, $\mathrm{i}Q\mathcal{F}f = \mathcal{F}g$). The converse is also true.

The relation $\varphi(P) = \mathcal{F}^{-1}\varphi(Q)\mathcal{F}$ gives in particular operators $P^n = \mathcal{F}^{-1}Q^n\mathcal{F}$.

2c4 Exercise. Let $n \in \{2, 3, \dots\}$.

(a) $P^n f$ is defined if and only if $P^{n-1}f$ is defined and belongs to D_P;

(b) in this case $P^n f = P(P^{n-1}f)$.

Prove it.

Hint: use 2b5.

For an infinitely differentiable compactly supported function f we have $(\mathrm{i}P)^n f = f^{(n)}$. It is tempting to conclude that

$$f(q+a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(q)\,, \quad \text{since} \quad \exp(\mathrm{i}aP) = \sum_{n=0}^{\infty} \frac{a^n}{n!} (\mathrm{i}P)^n\,,$$

but this conclusion is evidently wrong (unless $f = 0$). A series of unbounded operators is a more delicate matter!

2c5 Exercise. (a) $P^{-1}f$ is defined if and only if there exists $g \in D_P$ such that $Pg = f$;

(b) in this case such g is unique, and $P^{-1}f = g$.

Prove it.

Hint: use 2b6.

The Cayley transform of P (recall (2b4)) is the unitary operator $\varphi(P) = \mathcal{F}^{-1}\varphi(Q)\mathcal{F}$ where $\varphi : p \mapsto \frac{p-\mathrm{i}}{p+\mathrm{i}}$. It satisfies

$$(P - \mathrm{i}1\!\!1)f = (P + \mathrm{i}1\!\!1)Uf \quad \text{for } f \in D_P\,,$$

which means just $f' + f = g' - g$ where $g = Uf$, provided that f and g are nice enough (otherwise the derivatives are generalized). Can we calculate U more explicitly? Yes, we can! First we note that $\varphi = 1\!\!1 - 2\psi$, $\psi \in L_2$, $\psi : p \mapsto \frac{\mathrm{i}}{p+\mathrm{i}}$. Recalling Sect. 1h we observe that we can get $Uf = f - 2f * g$ if we find $g \in L_1$ such that $(2\pi)^{1/2}\mathcal{F}g = \psi$. Clearly, $g = (2\pi)^{-1/2}\mathcal{F}^{-1}\psi \in L_2$; but does g belong to L_1, and can we calculate it explicitly? Fortunately, such a function is well-known:

$$g(q) = \mathrm{e}^q 1\!\!1_{(-\infty,0)}(q)\,;$$
$$\int_{-\infty}^0 \mathrm{e}^q \mathrm{e}^{-\mathrm{i}pq}\,\mathrm{d}q = \int_{-\infty}^0 \mathrm{e}^{(1-\mathrm{i}p)q}\,\mathrm{d}q = \frac{1}{1-\mathrm{i}p} = \frac{\mathrm{i}}{p+\mathrm{i}}\,.$$

So,

$$Uf = f - 2f * g\,;$$
$$Uf : q \mapsto f(q) - 2\int_0^\infty \mathrm{e}^{-u} f(q+u)\,\mathrm{d}u\,.$$

2d Frequency bands, spectral projections

The operators Q and P have no eigenvectors but still have many invariant subspaces. The corresponding projections are instrumental in signal processing and quantum mechanics.

Indicator functions $\varphi = 1\!\!1_{(a,b)} \in L_\infty(\mathbb{R})$ satisfy $\varphi^2 = \varphi$ and $\overline{\varphi} = \varphi$, therefore the operators

$$E_{a,b} = E_{a,b}^{(Q)} = \varphi(Q) = 1\!\!1_{(a,b)}(Q)$$

are self-adjoint (that is, orthogonal) projections $L_2(\mathbb{R}) \to L_2(a,b) \subset L_2(\mathbb{R})$. The relation $1\!\!1_{(a,b)} + 1\!\!1_{(b,c)} = 1\!\!1_{(a,c)}$ in L_∞ (for $a < b < c$) implies the relation $E_{a,b} + E_{b,c} = E_{a,c}$ between operators, and the corresponding direct sum relation $L_2(a,b) \oplus L_2(b,c) = L_2(a,c)$ between subspaces. These subspaces are invariant under Q (and all $\varphi(Q)$). Note that

$$\|E_{a,b}^{(Q)} f\|^2 = \langle E_{a,b}^{(Q)} f, f\rangle = \int_a^b |f(q)|^2\,\mathrm{d}q\,.$$

In signal processing, $\|f\|^2$ is (proportional to) the energy of the signal f; $|f(t)|^2$ is the energy density at the time t; and $\langle E_{a,b}^{(Q)} f, f\rangle$ is the energy within the time interval (a,b).

In quantum mechanics, $|f(q)|^2$ is the probability density (at the point q) of the coordinate of a one-dimensional particle with the wave function f

($\|f\| = 1$ is required), and $\langle E_{a,b}^{(Q)} f, f \rangle$ is the probability of finding the particle within the spatial interval (a,b) (provided that the coordinate is measured).[1]

Accordingly, the operators

$$E_{a,b}^{(P)} = 1\!\!1_{(a,b)}(P) = \mathcal{F}^{-1} E_{a,b}^{(Q)} \mathcal{F}$$

are orthogonal projections satisfying $E_{a,b}^{(P)} + E_{b,c}^{(P)} = E_{a,c}^{(P)}$ (for $a < b < c$). The corresponding subspaces ("frequency bands") satisfy the direct sum relation, and are invariant under P (and all $\varphi(P)$).

2d1 Exercise.

$$\|E_{a,b}^{(P)} f\|^2 = \langle E_{a,b}^{(P)} f, f \rangle = \int_a^b |(\mathcal{F}f)(p)|^2 \, \mathrm{d}p \, .$$

Prove it.

Hint: $\mathcal{F}^{-1} = \mathcal{F}^*$.

In signal processing, $\|(\mathcal{F}f)(\omega)\|^2$ is the spectral density of the signal energy at the frequency ω; and $\langle E_{a,b}^{(P)} f, f \rangle$ is the energy within the frequency band (a,b).

In quantum mechanics, $|(\mathcal{F}f)(p)|^2$ is the probability density (at the point p) of the momentum of a one-dimensional particle with the wave function f ($\|f\| = 1$ is required), and $\langle E_{a,b}^{(P)} f, f \rangle$ is the probability of finding the momentum within the interval (a,b) (provided that the momentum is measured).[2]

2d2 Exercise. For every $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$,

$$E_{a,b}^{(P)} f = g_{a,b} * f \, , \quad \text{where}$$
$$g_{a,b}(q) = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}bq} - \mathrm{e}^{\mathrm{i}aq}}{q} \, .$$

Prove it.

Hint: $\mathcal{F}(g * f) = \dots$

Especially, $g_{-b,b}(q) = \frac{\sin bq}{\pi q}$.

Be careful: $g_{a,b}$ belongs to $L_2(\mathbb{R})$ but not $L_1(\mathbb{R})$. Nevertheless the convolution operator $f \mapsto g_{a,b} * f$ is well-defined on a dense set of functions f and extends by continuity to all $f \in L_2$.[3]

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[1] The *ideal* measurement of the coordinate is meant. Do not take it too seriously. It is rather a toy model of a quantum measurement. The infinite resolution is unfeasible.

[2] Once again, the *ideal* measurement of the momentum is meant...

[3] Which cannot be said about $|g_{a,b}(\cdot)|$...

2e List of formulas

Multiplication operators:

$$Qf : q \mapsto qf(q) \quad \text{for } f \in D_Q\,; \tag{2e1}$$

$$\varphi(Q)f = \varphi \cdot f : q \mapsto \varphi(q)f(q) \quad \text{for } f \in D_{\varphi(Q)}\,; \tag{2e2}$$

$$\exp(\mathrm{i}bQ) = V(b)\,; \tag{2e3}$$

$$Qf = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}b}\bigg|_{b=0} V(b)f \quad \text{for } f \in D_Q\,; \tag{2e4}$$

$$E^{(Q)}_{a,b} = 1\!\!1_{(a,b)}(Q)\,; \tag{2e5}$$

$$\|E^{(Q)}_{a,b} f\|^2 = \langle E^{(Q)}_{a,b} f, f\rangle = \int_a^b |f(q)|^2 \,\mathrm{d}q\,. \tag{2e6}$$

Operators commuting with shifts:

$$P = \mathcal{F}^{-1} Q \mathcal{F}\,; \tag{2e7}$$

$$Pf : q \mapsto -\mathrm{i}f'(q) \quad \text{for nice } f\,; \tag{2e8}$$

$$\varphi(P) = \mathcal{F}^{-1}\varphi(Q)\mathcal{F} : f \mapsto \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)\,; \tag{2e9}$$

$$\exp(\mathrm{i}aP) = U(a)\,; \tag{2e10}$$

$$Pf = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}a}\bigg|_{a=0} U(a)f \quad \text{for } f \in D_P\,; \tag{2e11}$$

$$E^{(P)}_{a,b} = 1\!\!1_{(a,b)}(P) = \mathcal{F}^{-1} E^{(Q)}_{a,b} \mathcal{F}\,; \tag{2e12}$$

$$\|E^{(P)}_{a,b} f\|^2 = \langle E^{(P)}_{a,b} f, f\rangle = \int_a^b |(\mathcal{F}f)(p)|^2 \,\mathrm{d}p\,; \tag{2e13}$$

$$E^{(P)}_{a,b} f = \left(q \mapsto \frac{1}{2\pi\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}bq} - \mathrm{e}^{\mathrm{i}aq}}{q} \right) * f \quad \text{for } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,. \tag{2e14}$$

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