
Let X be a reversible Markov chain, and let C be a non-empty subset of the state space S. Define the Markov chain Y on S by the transition matrix $Q = (q_{ij})$ where

$$q_{ij} = \begin{cases} \beta p_{ij} & \text{if } i \in C \text{ and } j \notin C, \\ p_{ij} & \text{otherwise,} \end{cases}$$

for $i \neq j$, and where β is a constant satisfying $0 < \beta < 1$. The diagonal terms q_{ii} are arranged so that Q is a stochastic matrix. Show that Y is reversible in equilibrium, and find its stationary distribution. Describe the situation in the limit as $\beta \downarrow 0$.

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We guess that the stationary distribution (q_i) of Y results from the stationary distribution (p_i) of X by

$$q_i = \begin{cases} ap_i & \text{if } i \in C, \\ bp_i & \text{otherwise} \end{cases}$$

for some constants $a, b > 0$. We have $p_i p_{ij} = p_j p_{ji}$ (the detailed balance for X), and we need $q_i q_{ij} = q_j q_{ji}$ for $i \neq j$ (the detailed balance for Y). If $i, j \in C$ then $q_{ij} = p_{ij}$, $q_{ji} = p_{ji}$, $q_i = ap_i$ and $q_j = ap_j$, therefore $q_i q_{ij} = ap_i p_{ij} = ap_j p_{ji} = q_j q_{ji}$. The case $i, j \notin C$ is similar. If $i \in C$ but $j \notin C$ then $q_{ij} = \beta p_{ij}$, $q_{ji} = p_{ji}$, $q_i = ap_i$ and $q_j = bp_j$; in order to get $ap_i \cdot \beta p_{ij} = bp_j \cdot p_{ji}$ we need

$$a\beta = b\,.$$

The last case, $i \notin C$ but $j \in C$. Here $q_{ij} = p_{ij}$, $q_{ji} = \beta p_{ji}$, $q_i = bp_i$ and $q_j = ap_j$; the equality $bp_i \cdot p_{ij} = ap_j \cdot \beta p_{ji}$ follows from $a\beta = b$. It remains to choose a, b such that $a\beta = b$ and $\sum q_i = 1$;

$$1 = \sum_i q_i = \sum_{i \in C} q_i + \sum_{i \notin C} q_i = a \sum_{i \in C} p_i + b \sum_{i \notin C} p_i = a \Bigl(\sum_{i \in C} p_i + \beta \sum_{i \notin C} p_i \Bigr)\,.$$

Finally,

$$a = \frac{1}{\sum_{i \in C} p_i + \beta \sum_{i \notin C} p_i}\,, \qquad b = \frac{\beta}{\sum_{i \in C} p_i + \beta \sum_{i \notin C} p_i}\,.$$

If β is small then b is small, thus, the distribution nearly concentrates on C. And no wonder: the exit from C becomes hard, while return to C remains easy.
