
A warehouse has a capacity to hold four items. If the warehouse is neither full nor empty, the number of items in the warehouse changes whenever a new item is produced or an item is sold. Suppose that (no matter when we look) the probability that the next event is "a new item is produced" is 2/3 and that the new event is a "sale" is 1/3. If there is currently one item in the warehouse, what is the probability that the warehouse will become full before it becomes empty.

. . . .

The states are $0, 1, 2, 3, 4$ and the transition probabilities are $p_{i,i+1} = 2/3$, $p_{i,i-1} = 1/3$. A 'one-step calculation' shows that the function

$$h(i) = P_i(V_4 < V_0)$$

satisfies the equation

$$h(i) = \frac{2}{3}h(i+1) + \frac{1}{3}h(i-1) \quad \text{for } 0 < i < 4$$

and the boundary conditions $h(0) = 0$, $h(4) = 1$. We have $\frac{2}{3}h(i+1) - \frac{2}{3}h(i) = \frac{1}{3}h(i) - \frac{1}{3}h(i-1)$, thus the differences $h(1) - h(0)$, $h(2) - h(1)$, $h(3) - h(2)$, $h(4) - h(3)$ are proportional to $8, 4, 2, 1$ and therefore equal to $8/15$, $4/15$, $2/15$, $1/15$, which means

$$h(1) = \frac{8}{15}, \quad h(2) = \frac{12}{15}, \quad h(3) = \frac{14}{15}.$$

The answer: $P_1(V_4 < V_0) = 8/15$.

Knight's random walk. If we represent our chessboard as $\{(i,j) : 1 \le i, j \le 8\}$ then a knight can move from (i,j) to any of eight squares $(i+2, j+1)$, $(i+2, j-1)$, $(i+1, j+2)$, $(i+1, j-2)$, $(i-1, j+2)$, $(i-1, j-2)$, $(i-2, j+1)$, or $(i-2, j-1)$, provided of course that they are on the chessboard. Let X_n be the sequence of squares that results if we pick one of knights legal moves at random. Find (a) the stationary distribution and (b) the expected number of moves to return to corner $(1,1)$ when we start there.

. . . .

This is a special case of the random walk on a (unoriented) graph; therefore the stationary probability $\pi(x)$ of a vertex x is proportional to the degree of x (that is, the number of neighbors). Thus,

$$\pi = \frac{1}{336}\begin{pmatrix} 2 & 3 & 4 & 4 & 4 & 4 & 3 & 2 \\ 3 & 4 & 6 & 6 & 6 & 6 & 4 & 3 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 3 & 4 & 6 & 6 & 6 & 6 & 4 & 3 \\ 2 & 3 & 4 & 4 & 4 & 4 & 3 & 2 \end{pmatrix}.$$

The average return time

$$\mathbb{E}_{(1,1)} T_{(1,1)} = \frac{1}{\pi_{(1,1)}} = \frac{1}{2/336} = 168.$$
