
An unfair fair game. Define random variables recursively by $X_0 = 1$ and for $n \ge 1$, X_n is chosen uniformly on $(0, X_{n-1})$. If we let $U_1, U_2, \dots$ be uniform on $(0,1)$, then we can write this sequence as $X_n = U_n U_{n-1} \dots U_1$. (a) Show that $M_n = 2^n X_n$ is a martingale. (b) Use the fact that $\ln X_n = \ln U_1 + \dots + \ln U_n$ to show that $(1/n) \ln X_n \to -1$. (c) Use (b) to conclude $M_n \to 0$, i.e., in this "fair" game our fortune always converges to 0 as time tends to ∞.

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(a) $M_n = (2U_1) \dots (2U_n)$ is a martingale, since $\mathbb{E}\,(2U_k) = 2 \cdot \frac{0+1}{2} = 1$ for all k.

(b) The random variable $(-\ln U_1)$ is distributed exponentially,

$$\mathbb{P}\big(-\ln U_1 > a\big) = \mathbb{P}\big(U_1 < \mathrm{e}^{-a}\big) = \mathrm{e}^{-a} \quad \text{for } a > 0\,,$$

therefore $\mathbb{E}\,(-\ln U_1) = 1$. Similarly, $\mathbb{E}\,\ln U_n = -1$ for all n. Using the strong law of large numbers we get

$$\frac{1}{n} \ln X_n = \frac{\ln U_1 + \dots + \ln U_n}{n} \to -1 \quad \text{a.s. as } n \to \infty\,.$$

Thus, $\frac{1}{n} \ln M_n = \frac{1}{n}(n \ln 2 + \ln X_n) \to -1 + \ln 2 < 0$, which implies $\ln M_n \to -\infty$ and $M_n \to 0$ a.s. as $n \to \infty$.

Modify the Polya's urn scheme so that we start with 5 green and 1 red. Use the stopping theorem to find an upper bound on the probability of $A = \{X_n > 1/2 \text{ for some } n \ge 0\}$.

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We know that X_n is a martingale. Considering the stopping time $\tau = \min\{n : X_n \ge 0.5\}$ and the stopped martingale $X_{\tau \wedge n}$ we get

$$\frac{1}{6} = X_0 = \mathbb{E}\,X_{\tau\wedge n} = \mathbb{E}\,(X_{\tau\wedge n}, \tau \le n) + \mathbb{E}\,(X_{\tau\wedge n}, \tau > n) \ge \frac{1}{2}\mathbb{P}\big(\tau \le n\big) + 0\,,$$

therefore $\mathbb{P}\big(\tau \le n\big) \le \frac{1}{3}$ for all n, and $\mathbb{P}\big(A\big) \le \mathbb{P}\big(\tau < \infty\big) = \lim_n \mathbb{P}\big(\tau \le n\big) \le \frac{1}{3}$.

Alice and Betty enter a beauty parlor simultaneously, Alice to get a manicure and Betty to get a haircut. Suppose the time for a manicure (haircut) is exponentially distributed with mean 20 (30) minutes. (a) What is the probability Alice gets done first? (b) What is the expected amount of time until Alice and Betty are both done?

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We have independent random variables S, T such that λS amd μT are distributed Exp(1). Using a hour as the unit of time we have $\mathbb{E}\,S = \frac{1}{3}$ and $\mathbb{E}\,T = \frac{1}{2}$, that is, $\lambda = 3$ and $\mu = 2$.

(a) $\mathbb{P}\big(S < T\big) = \frac{\lambda}{\lambda+\mu} = 0.6$.

(b) The shortest way to $\mathbb{E}\,\max(S,T)$ is the following trick: $\min(S,T) + \max(S,T) = S + T$, therefore $\mathbb{E}\,\min(S,T) + \mathbb{E}\,\max(S,T) = \mathbb{E}\,S + \mathbb{E}\,T = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$. However, $(\lambda+\mu)\min(S,T) \sim$ Exp(1), thus, $\mathbb{E}\,\min(S,T) = \frac{1}{\lambda+\mu} = \frac{1}{5}$. Therefore $\mathbb{E}\,\max(S,T) = \frac{5}{6} - \frac{1}{5} = \frac{19}{30}$ (that is, 38 min).
