
A light bulb has a lifetime that is exponential with a mean of 200 days. When it burns out a janitor replaces it immediately. In addition there is a handyman who comes at times of a Poisson process at rate 0.01 and replaces the bulb as "preventive maintenance". (a) How often is the bulb replaced? (b) In the long run what fraction of the replacements are due to failure?

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A new bulb either burns out after a time S or is replaced by the handyman after a time T; here $\frac{1}{200}S$ and $0.01T$ are independent $\mathrm{Exp}(1)$ random variables.

(a) $\big(\frac{1}{200}+0.01\big)\min(S,T) \sim \mathrm{Exp}(1)$, thus, a bulb is replaced in the mean after $1/0.015 = 66.7$ days.

(b) $\mathbb{P}\big(S<T\big) = 0.005/0.015 = 1/3$, thus, 1/3 of the replacements are due to failure.

Consider a Poisson process with rate λ and let L be the time of the last arrival in the interval $[0,t]$, with $L=0$ if there was no arrival. (a) Compute $\mathbb{E}\,(t-L)$. (b) What happens when we let $t\to\infty$ in the answer to (a)?

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The Poissonian random set (of arrivals) is invariant in distribution under time reversal $[0,t]\to[0,t]$, $s\mapsto t-s$. Therefore $t-L$ is distributed like $\min(T_1,t)$, where T_1 is the first arrival.

(a) $\mathbb{E}\,(t-L) = \mathbb{E}\,\min(T_1,t) = \mathbb{E}\,(T_1, T_1<t) + \mathbb{E}\,(t, T_1>t) = \mathbb{E}\,T_1 - \mathbb{E}\,(T_1-t, T_1>t) = \mathbb{E}\,T_1 - \mathbb{E}\,\big(\,T_1-t \,\big|\, T_1>t\,\big)\,\mathbb{P}\big(T_1>t\big)$; using the memoryless property we get $\mathbb{E}\,(t-L) = (\mathbb{E}\,T_1)\big(1-\mathbb{P}\big(T_1>t\big)\big) = (1/\lambda)(1-\mathrm{e}^{-\lambda t})$.

(b) $(1/\lambda)(1-\mathrm{e}^{-\lambda t}) \to 1/\lambda$ as $t\to\infty$.
