3 The Banach-Mazur game

3a Definitions and a simple fact

In Sect. 1a, for given $a_1, a_2, \dots \in \mathbb{R}$ we construct $[b_1, c_1] \supset [b_2, c_2] \supset \dots$ such that $a_n \notin [b_n, c_n]$. On stage n we do not need to know $a_{n+1}, \dots$; we need only a_n. Thus, the same idea leads to a game: Alice chooses $a_1 \in \mathbb{R}$; then Bob chooses $[b_1, c_1] \not\ni a_1$; then Alice chooses $a_2 \in \mathbb{R}$; then Bob chooses $[b_2, c_2] \subset [b_1, c_1]$, $[b_2, c_2] \not\ni a_2$; and so on. Similarly to Sect. 1a we wonder, what happens if Alice may choose a larger set (not only a singleton) at each step? Still, Bob is able to play provided that

$$\begin{gathered} A_0 \text{ is not dense in } \mathbb{R}\,, \quad \text{and} \\ A_{n+1} \text{ is not dense in } [b_n, c_n] \quad \text{for } n = 0, 1, 2, \dots \end{gathered}$$

where A_n is the set chosen by Alice on step n. And still, Bob wins; it means, he gets in the intersection a point outside $A_1 \cup A_2 \cup \dots$

The Banach-Mazur game[1] [2] is somewhat different. A set $A \subset \mathbb{R}$ is given. Alice chooses an interval U_1. Then Bob chooses a subinterval $V_1 \subset U_1$. Then Alice chooses $U_2 \subset V_1$; and so on. Finally, let the intersection of all these intervals be a singleton $\{x\}$; then Alice wins if $x \in A$ while Bob wins if $x \notin A$. Denoting by B the complement of A we may say: Bob wins if $x \in B$.

Some questions remain:

* what happens if the intersection is not a singleton?
* are the intervals open, close, or arbitrary? and what about sets more general than intervals?
* do Alice and Bob remember the past moves?

It appears that the answers are not important. The important question is, whether A is meager, comeager or neither.

For now we assume that

[1] The first infinite positional game of perfect information to be studied.

[2] In the literature it is usual to assign the first move to Bob and seek a winning strategy for Alice.

* all intervals must be open and nonempty;
* Bob forgets the past moves, and moreover, he forgets the number of these moves.

In this case, by definition,

* the *strategy* for Bob is a map σ from the set of such intervals to itself, satisfying $\sigma(U) \subset U$ for all U;
* a *run* of the game is $(U_n, V_n)_{n=1}^{\infty}$ such that $U_{n+1} \subset V_n \subset U_n$ for $n = 1, 2, \dots$
* the run is *compatible* with the strategy if $V_n = \sigma(U_n)$ for $n = 1, 2, \dots$;
* a strategy is *winning* for Bob if Bob wins all runs compatible with the strategy;

a question remains, when Bob wins a run. Note that Alice need not follow any strategy (for now).

We intend to prove that Bob wins whenever A is meager. In order to make this claim stronger we also assume that Bob is responsible for the singleton in the intersection. That is, by definition,

* Bob *wins* the run if $\cap_n U_n = \cap_n V_n = \{x\}$ for some $x \in B$

(and only in this case). Of course, the equality $\cap_n U_n = \cap_n V_n$ holds for every run.

3a1 Proposition. If B is comeager then Bob has a winning strategy.

Note that Bob wins even if he is memoryless and responsible for the singleton (the worst case). The more so he wins in all more favorable cases.

It is simpler to prove it for a bit more favorable case: Bob knows n (the number of the move). In this case a strategy is $(\sigma_n)_n$ and compatibility is $V_n = \sigma_n(U_n)$. We choose σ_n such that

$$\begin{gathered} \mathrm{Cl}(\sigma_n(U)) \subset U\,, \\ |\sigma_n(U)| \le 2^{-n}\,, \\ \sigma_n(U) \cap A_n = \emptyset\,; \end{gathered}$$

here $|\dots|$ is the length of the interval, and A_n are nowhere dense sets such that $A \subset \cup_n A_n$. Clearly, such σ_n exist and are a winning strategy.

Now, the worst case.

Proof of 3a1. We choose σ such that

$$\begin{gathered} \mathrm{Cl}(\sigma(U)) \subset U\,, \\ |\sigma(U)| \le \frac{1}{2} \min(1, |U|)\,, \\ \sigma(U) \cap A_n = \emptyset \quad \text{whenever } |U| \le 2^{-n}\,. \end{gathered}$$

This is evidently possible, and implies for every compatible run $(U_n, V_n)_n$

$$\begin{aligned} |V_n| &\le 2^{-n} \quad \text{for } n = 1, 2, \dots ; \\ |U_n| &\le 2^{-(n-1)} \quad \text{for } n = 2, 3, \dots ; \\ V_n \cap A_{n-1} &= \emptyset \quad \text{for } n = 2, 3, \dots ; \end{aligned}$$

it follows that Bob wins the run. □

Why just intervals on $\mathbb{R}$? We may consider rather general subsets of a metric space.

Assume that (X, ρ) is a metric space and $M_{\mathrm{A}}, M_{\mathrm{B}}$ ("possible moves of Alice", "of Bob") are given sets of subsets of X satisfying

$$\forall U \in M_{\mathrm{A}}\ \exists V \in M_{\mathrm{B}}\ \ V \subset U\,, \tag{3a2}$$

$$\forall V \in M_{\mathrm{B}}\ \exists U \in M_{\mathrm{A}}\ \ U \subset V\,. \tag{3a3}$$

(In particular, both may consist of all nonempty open sets in X; this is the most usual choice.) A run of the game is, by definition, $(U_n, V_n)_n$ such that $U_n \in M_{\mathrm{A}}$, $V_n \in M_{\mathrm{B}}$ and $U_{n+1} \subset V_n \subset U_n$ for $n = 1, 2, \dots$

We do not want to restrict ourselves to compact spaces (indeed, $\mathbb{R}$ is not compact) and use completeness instead.

3a4 Exercise. The following conditions on a metric space (X, ρ) are equivalent:

(a) Every Cauchy sequence $(x_n)_n$ in X converges; that is,

$$\inf_n \sup_k \rho(x_n, x_{n+k}) = 0 \quad \Longrightarrow \quad \exists x\ \ \rho(x_n, x) \to 0\,.$$

(b) If closed sets $F_n \subset X$ satisfy $F_1 \supset F_2 \supset \dots$ and $\operatorname{diam} F_n \to 0$ then $\cap_n F_n \ne \emptyset$; here $\operatorname{diam} F_n = \sup_{x,y \in F_n} \rho(x, y)$.

(c) X is closed in every including metric space; that is, if (Y, ρ_1) is a metric space such that $X \subset Y$ and $\rho(x_1, x_2) = \rho_1(x_1, x_2)$ for all $x_1, x_2 \in X$ then X is closed in Y.

Prove it.

3a5 Definition. (a) A metric space (X, ρ) is *complete* if it satisfies the equivalent conditions 3a4(a,b,c).

(b) A metrizable space (X, R) is *completely metrizable* if (X, ρ) is complete for some $\rho \in R$.

(c) A metrizable space is *separable* if some sequence is dense.

(d) A metrizable space (X, R) is *Polish*[1] if it is completely metrizable and separable.

[1] Sierpiński, Kuratowki, Tarski...

However, “Polish metric space” is ambiguous; for some authors it is “separable complete metric space”, while others mean completeness in some equivalent metric.

3a6 Exercise. A metrizable space is separable if and only if there exists a *countable base,* that is, a sequence $(U_n)_n$ of open sets such that $U = \cup_{n:U_n \subset U} U_n$ for every open set U.

Prove it.

3a7 Exercise. A subspace of a separable space is separable.

Prove it.

3a8 Exercise. (a) A compact space is separable.

(b) A compact space is complete in every compatible metric.

Prove it.

Thus, a compact space is Polish.

The space $[0, 1]$ is compact; $(0, 1)$ is not compact, and not complete, but still Polish (being homeomorphic to $\mathbb{R}$).

3a9 Proposition. Let X be completely metrizable, M_A satisfy

$$\text{(a)} \qquad \forall U \in M_\mathrm{A} \;\; \mathrm{Int}(U) \ne \emptyset$$

and M_B satisfy

$$\text{(b)} \qquad \text{for every nonempty open } G \subset X \,, \quad \exists V \in M_\mathrm{B} \;\; V \subset G \,.$$

If B is comeager then Bob has a winning strategy.

Similarly to 3a1, the worst case is meant. Of course, (3a2), (3a3) are still assumed. It follows from (3a3) and 3a9(a) that $\forall V \in M_\mathrm{B} \;\; \mathrm{Int}(V) \ne \emptyset$. Still, M_A and M_B may consist of all nonempty open sets.

3a10 Exercise. Prove 3a9.

3a11 Theorem (Baire)**.** Let X be a completely metrizable space. If $A_1, A_2, \dots \subset X$ are nowhere dense then $\mathrm{Int}(\cup_n A_n) = \emptyset$.

Note that 1d3 and 1d6 are special cases of 3a11.

3a12 Exercise. (a) Deduce 3a11 from 3a9.

(b) Give another proof of 3a11, simple and free of games.

3b The converse holds but is not simple

Now we want to prove the converse to 3a1, 3a9: if B is not comeager then Bob has no winning strategy. This is easy to see if A is comeager, since then Alice has a winning strategy. The question is, what happens if A and B are neither meager nor comeager.

We assume that Bob has a winning strategy and want to prove that B is somehow large (ultimately, comeager). For now we deal with open intervals in $\mathbb{R}$. We know that B is not meager, therefore uncountable. Can we prove (at least) that B is of cardinality continuum?

Consider two disjoint intervals; denote them $U_1(0), U_1(1)$. Alice may choose any one of them, $U_1 = U_1(b_1)$, $b_1 \in \{0,1\}$. Bob chooses $V_1 = V_1(b_1) \subset U_1(b_1)$ according to his winning strategy. Consider two disjoint intervals of both:

$$U_2(0,0), U_2(0,1) \subset V_1(0)\,, \quad U_2(1,0), U_2(1,1) \subset V_1(1)\,.$$

Alice may choose any one of them, $U_2 = U_2(b_1, b_2)$, $b_2 \in \{0,1\}$. And so on. In all cases Bob is guaranteed to win; it means that

$$\bigcap_n U_n(b_1,\dots,b_n) = \{x(b_1,b_2,\dots)\}\,, \quad x(b_1,b_2,\dots) \in B$$

for all $(b_n)_n \in \{0,1\}^\infty$. These points are pairwise distinct (since the intervals are disjoint...); thus, B is of cardinality continuum.

It is easy to see that all these $x(b)$ for $b \in \{0,1\}^\infty$ are a set C homeomorphic to the Cantor set. Yes, it is of cardinality continuum, but not at all comeager; it is nowhere dense. Can we improve the trick? Let us try to understand it better.

We consider a tree T_2, a binary[1] subtree of the much larger tree T of all *legal positions* of the game; these are $(U_1, V_1, \dots, U_n, V_n)$ and $(U_1, V_1, \dots, U_n, V_n, U_{n+1})$ in general, but we restrict ourselves to $U_k(b_1,\dots,b_k)$, $V_k(b_1,\dots,b_k)$. Infinite branches of T_2 are the considered runs $(U_n(b_1,\dots,b_n), V_n(b_1,\dots,b_n))_n = (U_n(b[1:n]), V_n(b[1:n]))_n$, $b \in \{0,1\}^\infty$. We note that

$$\bigcap_n U_n(b[1:n]) = \bigcap_n V_n(b[1:n]) = \{x(b)\}\,,$$

$$C = \bigcup_b \bigcap_n V_n(b[1:n])\,,$$

$$\bigcup_b \bigcap_n V_n(b[1:n]) = \bigcap_n \underbrace{\bigcup_b V_n(b[1:n])}_{G_n(T_2)}\,. \tag{3b1}$$

[1] Not quite binary; Alice has a binary choice on each move, but Bob follows a strategy.

Thus, C is the intersection of a sequence of open sets $G_n(T_2)$. These sets are not dense; what a pity…

Can we use a larger subtree? The whole tree T does not fit, since Bob follows a strategy σ. Consider the corresponding subtree T_σ. The set

$$G_n(T_\sigma) = \bigcup_{(U_1,V_1,\dots,U_n,V_n)\in T_\sigma} V_n$$

is dense (just because $V_n \subset U_1$ and U_1 is arbitrary). Nice; but what about (3b1)?

If a point x belongs to $G_n(T_\sigma)$ for all n, it means that $x \in V_n$ for some branch, for each n; but the branch may depend on n, this is the problem![1]

Why does (3b1) hold for T_2? Since the intervals are disjoint… Namely, for every n the 2^n sets $U_n(b_1,\dots,b_n)$ are pairwise disjoint; and therefore the sets $V_n(b_1,\dots,b_n)$ are pairwise disjoint, too.

We need disjointedness; but we do not really need so much disjointedness! It would be enough to have the disjointedness for infinitely many n (but not all n). Likewise, it would be enough to have the disjointedness for V_n (but not U_n). The latter appears to be the key idea!

Now we are in position to prove that B is comeager if Bob has a winning strategy. In order to make this claim stronger we consider the case best for Bob: he remembers the past moves, and Alice is responsible for the singleton. That is, we define:

- a strategy for Bob is a sequence $(\sigma_n)_n$ of maps σ_n from legal positions $(U_1, V_1, \dots, U_n)$ to M_{B};
- $(U_1, V_1, \dots, U_n)$ is a legal position if $U_1, \dots, U_n \in M_{\mathrm{A}}$, $V_1, \dots, V_{n-1} \in M_{\mathrm{B}}$ and $U_1 \supset V_1 \supset \dots \supset U_{n-1} \supset V_{n-1} \supset U_n$;
- a run is compatible with the strategy if $V_n = \sigma_n(U_1, V_1, \dots, U_n)$ for $n = 1, 2, \dots$;
- Alice wins the run if $\cap_n U_n = \cap_n V_n = \{x\}$ for some $x \in A$; otherwise Bob wins the run.

3b2 Proposition. Let X be Polish, M_{B} satisfy

(a) $$\forall V \in M_{\mathrm{B}} \;\; \mathrm{Int}(V) \ne \emptyset$$

and M_{A} satisfy

(b) $$\text{for every nonempty open } G \subset X\,, \quad \exists U \in M_{\mathrm{A}} \;\; U \subset G\,.$$

If B is not comeager then Bob has no winning strategy.

[1] König's lemma does not help, since Alice has infinitely many possible moves…

Of course, (3a2), (3a3) are still assumed (and therefore 3b2(a) is equivalent to 3a9(a)).

A *weak basis* for a topological space is a set of nonempty open sets such that every nonempty open set contains (at least) one of them.[1]

Given a nonempty open $G \subset X$ and n, we introduce

$$W_n(G) = \{\operatorname{Int} \sigma_n(U_1, V_1, \dots, U_n) : U_n \in M_{\mathrm{A}}, \operatorname{Cl}(U_n) \subset G, \operatorname{diam}(U_n) \le 2^{-n}\} \,.$$

3b3 Exercise. Prove that $W_n(G)$ is a weak basis for G.

3b4 Exercise. Let W be a weak basis for a separable metrizable space X. Then there exists a finite or infinite sequence $(w_n)_n$ of $w_n \in W$ such that all w_n are pairwise disjoint, and $\cup_n w_n$ is dense.

Prove it.

Usually it is an infinite sequence; for simplicity I consider only this case. The finite case is simpler, but complicates notations.

3b5 Corollary. Given a nonempty open $G \subset X$ and n, we get $U_k \in M_{\mathrm{A}}$ such that $\operatorname{Cl}(U_k) \subset G$, $\operatorname{diam} U_k \le 2^{-n}$, sets $\operatorname{Int} \sigma_n(U_1), \operatorname{Int} \sigma_n(U_2), \dots$ are pairwise disjoint, and their union is dense in G.

3b6 Exercise. Let G, G_k and $G_{k,l}$ be open sets ($k, l = 1, 2, \dots$) such that $\cup_k G_k$ is dense in G and for each k, $\cup_l G_{k,l}$ is dense in G_k. Then $\cup_{k,l} G_{k,l}$ is dense in G.

Prove it.

Proof of Prop. 3b2. Assume the contrary: Bob has a winning strategy $\sigma = (\sigma_n)_n$. Applying 3b5 to $G = X$ and $n = 1$ we get $U_k \in M_{\mathrm{A}}$ such that $\operatorname{diam} U_k \le 1/2$, $\cup_k \operatorname{Int} V_k$ is dense in X, and they are disjoint; here $V_k = \sigma_1(U_k)$.

Similarly, for each k we apply 3b5 to $G = \operatorname{Int} V_k$ and $n = 2$ and get $U_{k,l} \in M_{\mathrm{A}}$ such that $\operatorname{Cl}(U_{k,l}) \subset \operatorname{Int} V_k$, $\operatorname{diam} U_{k,l} \le 1/4$, $\cup_l \operatorname{Int} V_{k,l}$ is dense in $\operatorname{Int} V_k$, and they are disjoint; here $V_{k,l} = \sigma_2(U_{k,l})$. By 3b6, $\uplus_{k,l} \operatorname{Int} V_{k,l}$ is dense in X.

Continuing this way we get $U_n(a_1, \dots, a_n)$ and $V_n(a_1, \dots, a_n)$ such that $V_n(a_1, \dots, a_n) = \sigma_n(U_n(a_1, \dots, a_n)), \operatorname{Cl} U_{n+1}(a_1, \dots, a_{n+1}) \subset \operatorname{Int} V_n(a_1, \dots, a_n)$, $\operatorname{diam} U_n(a_1, \dots, a_n) \le 2^{-n}$, $\operatorname{Int} V_{n+1}(a_1, \dots, a_n, a') \cap \operatorname{Int} V_{n+1}(a_1, \dots, a_n, a'') = \emptyset$ for $a' \ne a''$, and $\cup_{a_1, \dots, a_n} \operatorname{Int} V_n(a_1, \dots, a_n)$ is dense in X. It follows that $\big(U_n(a[1:n]), V_n(a[1:n])\big)_n$ is a run compatible with σ for every $a \in \{1, 2, \dots\}^\infty$, and $\cap_n U_n(a[1:n]) = \cap_n V_n(a[1:n]) = \{x\}$ for some $x \in X$.

[1] Sect. 8.G in: A.S. Kechris, "Classical descriptive set theory", Springer 1995. Probably not a standard terminology except (maybe) descriptive set theory.

The set

$$C = \bigcap_n \bigcup_{a_1,\dots,a_n} \operatorname{Int} V_n(a_1,\dots,a_n)$$

is comeager. It remains to prove that $C \subset B$.

Let $x \in C$; we have to prove that $x \in B$. We note that $x \in \cup_{a_1} \operatorname{Int} V_1(a_1)$ and take $\tilde a_1$ such that $x \in \operatorname{Int} V_1(\tilde a_1)$; such $\tilde a_1$ is unique, since the sets are disjoint. Further, $x \in \cup_{a_1.a_2} \operatorname{Int} V_2(a_1,a_2)$; we take $a_1, \tilde a_2$ such that $x \in \operatorname{Int} V_2(a_1,\tilde a_2)$ and observe that $a_1 = \tilde a_1$ since $x \in \operatorname{Int} V_2(a_1,\tilde a_2) \subset \operatorname{Int} V_1(a_1)$. And so on; $x \in \operatorname{Int} V_n(\tilde a[1:n])$ for all n. Bob is guaranteed to win the run $\big(U_n(\tilde a[1:n]), V_n(\tilde a[1:n])\big)$, therefore $x \in B$. □

We see that the winning sets for Bob are exactly the comeager sets. This is an interesting characterization (equivalent definition) of "meager" and "comeager", free of "nowhere dense".

What about sets of full Lebesgue measure (on $\mathbb{R}$)? These could not be characterized via the Banach-Mazur game, since this game is evidently invariant under homeomorphisms of $\mathbb{R}$ (to itself), while Lebesgue measure is not, and moreover, the σ-ideal of null sets is not.

A "more quantitative" game may be designed as follows: Alice and Bob choose intervals (not just open sets) and each interval must be twice shorter than the previous interval.[1] This is a special case of so-called Schmidt's game[2] used in Diophantine approximations, ergodic theory etc. It is related to some classes of measures,[3] but fails to characterize the sets of full Lebesgue measure. The complements of the winning sets are a σ-ideal; but this σ-ideal appears to contain a set that is both comeager and of full Lebesgue measure!

We return to the Banach-Mazur game. What are the winning sets for Alice? Such a set need not be "comeager everywhere", it is enough to be "comeager somewhere".

3b7 Exercise. Let X be a metrizable space, $U \subset X$, $U \ne \emptyset$.

(a) If $A \subset U$ is nowhere dense in U (treated as another metrizable space) then A is nowhere dense in X;

(b) If $A \subset U$ is meager in U then A is meager in X;

(c) if U is open and $A \subset U$ is nowhere dense in X then A is nowhere dense in U;

(d) if U is open and $A \subset U$ is meager in X then A is meager in U;

[1] Another version: they choose binary digits, one after another. The results are similar.

[2] W.M. Schmidt (1966) "On badly approximable numbers and certain games", Trans. AMS **123**, 178–199.

[3] R. Broderick, Y. Bugeaud, L. Fishman, D. Kleinbock, B. Weiss (2010) "Schmidt's game, fractals, and numbers normal to no base", Math. Res. Lett. **17**:2, 307–321.

(e) items (c), (d) may fail if U is not open.
Prove it.

For an open $U \subset X$ we see that $A \subset U$ is meager in U if and only if A is meager in X. Thus, $A \subset U$ is comeager in U if and only if $U \setminus A$ is meager in X.

3b8 Definition. [1] Let X be a metrizable space, $U \subset X$ a nonempty open set, and $A \subset X$.

* A is meager in U, if $A \cap U$ is meager;
* A is comeager in U, if $U \setminus A$ is meager (equivalently: $A \cap U$ is comeager in U);
* if A is comeager in U, we say that A holds generically in U or that U *forces* A, and write $U \Vdash A$.

3b9 Exercise. A set is winning for Alice if and only if it is forced by some nonempty open set.

Prove it.

Do you think that a winning strategy (either for Alice or for Bob) is guaranteed to exist in all cases?

[1] Kechris, Sect. 8.G.

Hints to exercises

3a4: (c) either use completion, or add to X a single (limit) point.

3a6: "only if": use $\frac{1}{m}$-neighborhood of x_n.

3a7: use 3a6; or alternatively, take $y_n \in Y$ such that $\rho(y_n, x_n) \le 2 \inf_{y \in Y} \rho(y, x_n)$.

3a8: (a) if a space is not separable then $\inf\{\rho(x_m, x_n) : m \ne n\} > 0$ for some $(x_n)_n$.

3a10: similar to 3a1.

3b4: use 3a6.

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