7 About Egorov's and Lusin's theorems

7a About Severini-Egorov theorem

For every sequence of measurable functions $f_n : [0,1] \to \mathbb{R}$ there exists a sequence of numbers $\varepsilon_n > 0$ such that $\varepsilon_n f_n \to 0$ almost everywhere. This fact follows easily from the first Borel-Cantelli lemma; we take ε_n such that $\sum_n m\{x : |\varepsilon_n f_n| > \frac{1}{n}\} < \infty$, then $|\varepsilon_n f_n| \le 1/n$ for $n > N(x)$.

"Almost" in "almost everywhere" cannot be dropped. Here is a counterexample:

(7a1)
$$f_n(x) = \frac{1}{\operatorname{dist}(2^n x, \mathbb{Z})}$$

(if $2^n x \notin \mathbb{Z}$; otherwise $f_n(x) = 0$).[1] Taking $x = \sum_k 2^{-n_k}$ with $n_{k+1} - n_k \ge 2$ we get $f_{n_k}(x) \ge 2^{n_{k+1}-n_k-1}$ (think, why).

7a2 Exercise. If $f_n : [0,1] \to \mathbb{R}$ satisfy $f_n \to \infty$ almost everywhere then there exist c_n, C_n such that $0 < c_n < C_n < \infty$, $c_n \uparrow \infty$, $C_n \uparrow \infty$ and $c_n \ll f_n \ll C_n$ (that is, $f_n/c_n \to \infty$ and $f_n/C_n \to 0$) almost everywhere.

Prove it. Do the same for $f_n \to 0$.

On the other hand, given c_n, C_n we may take, say, $f_n = c_n$ on $(0, 1/2)$ and $f_n = C_n$ on $(1/2, 1)$, violating both relations $c_n \ll f_n$, $f_n \ll C_n$.

Thus, given a sequence of random variables that tends to infinity almost surely, we always can say something (but maybe not too much) about the rate of growth. The same holds for convergence to 0, of course.

Having $|f_n| \ll \varepsilon_n$ almost everywhere we also have $m\{x : \forall k \;\; |f_{n+k}(x)| \le \varepsilon_n\} \to 1$ as $n \to \infty$, which leads to the following.

7a3 Theorem. (Severini-Egorov) If $f, f_n : [0,1] \to \mathbb{R}$ are measurable functions such that $f_n \to f$ almost everywhere then for every $\varepsilon > 0$ there exists a measurable $A \subset [0,1]$ such that $m(A) \ge 1 - \varepsilon$ and $f_n \to f$ uniformly on A.

[1] Or $f_n(x) = \tan 2^n x$, if you like.

By regularity (recall (6a1)), A may be chosen to be compact.

7a4 Corollary. There exist $A_n \subset [0,1]$, $A_n \uparrow$, such that $\cup_n A_n$ is of full measure and $f_n \to f$ uniformly on each A_n.

Nothing like that holds in the topological approach. Recall for example the random walk conditioned to stay positive[1] (Sect. 2b); there, $n - x(n) \uparrow \infty$ for quasi all $x \in X^+$, but nothing can be said about the rate of convergence (lower bound, I mean; the upper bound $n - x(n) \le n$ is trivial). According to 2b3, $a_n = \log\log\log n$ are not a lower bound.

7a5 Exercise. For $f_n : X^+ \to \mathbb{R}$ defined by

$$f_n(x) = \frac{1}{n+1-x(n)}$$

prove that $f_n \downarrow 0$ quasi-everywhere, but the convergence can be uniform on $A \subset X^+$ only if A is nowhere dense.

In contrast to 7a4, if $A_n \subset X^+$ are such that $f_n \to 0$ uniformly on each A_n then $\cup_n A_n$ cannot be comeager, and moreover, must be meager.

7a6 Exercise. Let $(x_n)_n$ be the random walk.[2]

(a) Prove that $\{n : x(n) = 0\}$ is infinite for quasi all $x \in X$;

(b) defining $n_1(x) < n_2(x) < \dots$ by $\{n : x_n = 0\} = \{n_1(x), n_2(x), \dots\}$ prove that for every $(a_k)_k$ the set $\{x : \forall k \;\; n_{k+1}(x) - n_k(x) \le a_k\}$ is nowhere dense;

(c) continuing (b) prove that

$$\limsup_n \frac{n_{k+1}(x) - n_k(x)}{a_k} = \infty \quad \text{for quasi all } x \in X\,.$$

Here, nothing can be said on the rate of growth. And the same holds for the functions $f_n : [0,1] \to \mathbb{R}$ defined by (7a1): for every $(a_n)_n$,

$$\limsup_n \frac{f_n(x)}{a_n} = \infty \quad \text{for quasi all } x \in \mathbb{R}$$

by 2b1 (for $c = \infty$) or 5b9.

[1] Though, the same holds for the unconditioned random walk.

[2] Unconditioned. Though, the same holds under the condition.

7b About Lusin's theorem

7b1 Theorem. (Lusin) If $f : [0,1] \to \mathbb{R}$ is a measurable function then for every $\varepsilon > 0$ there exists a measurable $A \subset [0,1]$ such that $m(A) \ge 1 - \varepsilon$ and the restriction $f|_A$ is continuous (on A).

Once again, by regularity, A may be chosen to be compact.

Do not think that the whole f is continuous on A. (Try $\mathbb{1}_\mathbb{Q}$.)

For $f = \mathbb{1}_\mathbb{Q}$, $f|_A$ is continuous for some A of full measure, but this is not the general case, as we'll see soon.

7b2 Example. There exists a measurable function $f : [0,1] \to \mathbb{R}$ such that

$$m\{x \in (a,b) : f(x) \in (c,d)\} > 0$$

for all $(a,b) \subset [0,1]$ and $(c,d) \subset \mathbb{R}$.

Here is why. We take a sequence $(y_n)_n$ dense in $\mathbb{R}$ and a sequence of intervals $I_n = (x_n - 3^{-n}, x_n + 3^{-n}) \subset [0,1]$ such that $(x_n)_n$ is dense in $[0,1]$. Almost every $x \in [0,1]$ belongs only to finitely many intervals I_n; we take $f(x) = y_n$ for all $x \in I_n \setminus (I_{n+1} \cup I_{n+2} \cup \dots)$ and, say, $f(x) = 0$ for $x \in [0,1] \setminus \cup_n I_n$.

7b3 Exercise. Prove that this function is indeed an example.

7b4 Exercise. Prove that Theorem 7b1 generally fails for $\varepsilon = 0$.

Theorem 7a3 fails for $\varepsilon = 0$ evidently (just try $f_n(x) = x^n$).

Proof of Theorem 7b1. Continuous functions being dense (in measure) among (equivalence classes of) measurable functions, we take continuous $f_n : [0,1] \to \mathbb{R}$ such that $f_n \to f$ almost everywhere. Theorem 7a3 gives measurable $A \subset [0,1]$ such that $m(A) \ge 1 - \varepsilon$ and $f_n \to f$ uniformly on A. It follows that $f|_A$ is continuous. □

The topological approach contains nothing like Theorem 7a3 and nevertheless it contains a counterpart of Theorem 7b1, and moreover, of "Theorem 7b1 with $\varepsilon = 0$".

7b5 Definition. Let X, Y be metrizable spaces. A map $f : X \to Y$ has the Baire property[1] (symbolically, $f \in \mathrm{BP}(X \to Y)$) if $f^{-1}(V) \in \mathrm{BP}(X)$ for all open $V \subset Y$.

[1] "Baire measurable" according to Kechris (Sect. 8.I) which may be misleading (as noted in Sect. 6c).

7b6 Theorem. Let X be completely metrizable, Y separable, and $f \in \mathrm{BP}(X \to Y)$. Then there exists a comeager $A \subset X$ such that $f|_A$ is continuous (on A).

Proof. We take a countable basis $(V_n)_n$ of Y and open $U_n \subset X$ such that sets $U_n \triangle f^{-1}(V_n)$ are meager. The set $A = X \setminus \cup_n (U_n \triangle f^{-1}(V_n))$ is comeager, and $(f|_A)^{-1}(V_n) = U_n \cap A$. □

The converse is also true: if $f|_A$ is continuous for some comeager $A \subset X$ then $f \in \mathrm{BP}(X \to Y)$, since for every open $V \subset Y$ there exists open $U \subset X$ such that $f^{-1}(V) = U \cap A$ and therefore $f^{-1}(V) \in \mathrm{BP}(X)$.

Can we prove Lusin's theorem 7b1 similarly to 7b6, avoiding 7a3? Yes, we can.

Proof of Theorem 7b1 (again). We take a countable basis $(V_n)_n$ of $\mathbb{R}$ and, using regularity (6a1), open $U_n \subset [0,1]$ such that $f^{-1}(V_n) \subset U_n$ and $m\big(U_n \setminus f^{-1}(V_n)\big) \le 2^{-n}\varepsilon$. The set $A = [0,1] \setminus \cup_n (U_n \setminus f^{-1}(V_n))$ satisfies $m(A) \ge 1 - \varepsilon$, and $(f|_A)^{-1}(V_n) = U_n \cap A$. □

7c About measurable functions

If $f \in \mathrm{BP}(X \to Y)$ and $B \in \mathrm{BP}(Y)$, it does not imply $f^{-1}(B) \in \mathrm{BP}(X)$. Moreover, this can fail for a nowhere dense B. Here is why.

7c1 Exercise. There exists a one-to-one $f \in \mathrm{BP}([0,1) \to [0,1])$ such that $f([0,1))$ is contained in the Cantor set.

Prove it.

Let $V \subset [0,1)$ be a Vitali set and $B = f(V)$, then B is nowhere dense, but $f^{-1}(B) = V \notin \mathrm{BP}([0,1])$.

A set endowed with a σ-algebra (of subsets) is called *measurable space;* subsets belonging to the σ-algebra are called measurable. A map from one measurable space to another is called measurable if the inverse image of every measurable set is measurable.

Several useful σ-algebras are defined on every metrizable space X. First, the Borel σ-algebra $\mathcal{B}(X)$ generated by open sets. Second, $\mathrm{BP}(X)$ generated by open sets and nowhere dense sets.[1]

Let X be a measurable space and Y a metrizable space. The following two conditions on a map $f : X \to Y$ are equivalent:

∗ $f^{-1}(V)$ is measurable (in X) for every open $V \subset Y$;

[1]Third, universally measurable sets, etc.; these will not be used here.

∗ $f^{-1}(B)$ is measurable (in X) for every Borel $B \subset Y$; in other words, f is measurable from X to $(Y, \mathcal{B}(Y))$.

Given two metrizable spaces X, Y, we may endow X with $\mathcal{B}(X)$ and get so-called Borel maps (or Borel measurable maps) $f : X \to Y$. We also may endow X with $\mathrm{BP}(X)$ and get $f \in \mathrm{BP}(X \to Y)$. In addition, when $X = \mathbb{R}$, we may endow it with the Lebesgue σ-algebra and get Lebesgue measurable maps $\mathbb{R} \to Y$.

In all cases, on Y the relevant sets are Borel measurable. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $B \subset \mathbb{R}$ is Lebesgue measurable, it does not imply Lebesgue measurability of $f^{-1}(B)$. This phenomenon was already noted for BP (see 7c1 and around it).

7c2 Exercise. Let X be a measurable space and $f_n : X \to [-\infty, +\infty]$ measurable functions.

(a) If $f : X \to [-\infty, +\infty]$, $f(\cdot) = \sup_n f_n(\cdot)$, then f is measurable.

(b) If $f : X \to [-\infty, +\infty]$, $f(\cdot) = \limsup_n f_n(\cdot)$, then f is measurable.

(c) If $f : X \to [-\infty, +\infty]$, $f_n(\cdot) \to f(\cdot)$ as $n \to \infty$, then f is measurable.

Prove it.

7c3 Exercise. Let X be a measurable space, Y a separable metrizable space, $f_n : X \to Y$ measurable functions, and $f_n(\cdot) \to f(\cdot)$ as $n \to \infty$, then f is measurable.

Prove it.

The latter fact applies to Borel functions, as well as functions with Baire property (and also to Lebesgue measurable functions on $\mathbb{R}$).

Having on X also a σ-ideal (in addition to a σ-algebra) we get an equivalence relation between measurable functions, and a quotient set (of equivalence classes). For indicator functions $X \to \{0, 1\}$ these are equivalence classes of measurable sets. Two such cases were treated in Sect. 6d: $\mathcal{A}_m/\overset{m}{\sim}$ and $\mathrm{BP}/\!\sim$.

A measurable function $f : X \to \mathbb{R}$ may be described by a family $(A_t)_{t\in\mathbb{R}}$ of measurable sets $A_t = f^{-1}\big((-\infty, t]\big) \subset X$ satisfying

$$
\begin{gathered}
s \le t \quad \Longrightarrow \quad A_s \subset A_t \,, \\
\bigcap_t A_t = \emptyset \,, \quad \bigcup_t A_t = X \,, \\
\forall t \in \mathbb{R} \quad \bigcap_{\varepsilon > 0} A_{t+\varepsilon} = A_t \,.
\end{gathered}
$$

Every such family $(A_t)_t$ corresponds to some (evidently unique) f; just $f(x) = \min\{t \in \mathbb{R} : x \in A_t\}$.

Similarly, an equivalence class of measurable functions may be described by a family of equivalence classes of measurable sets. In order to restore $[f]$ from $([A_t])_t$ choose $A_t \in [A_t]$ for $t \in \mathbb{Q}$ and let $f(t) = \inf\{t \in \mathbb{Q} : x \in A_t\}$.

An equivalence class of measurable maps $f : X \to Y$ (Y being a separable metrizable space) may be described by a family $\big([A_B]\big)_B$ of equivalence classes of measurable sets, indexed by all Borel sets $B \subset Y$, satisfying

$$[A_\emptyset] = [\emptyset] , \quad [A_Y] = [X] ,$$
$$[A_{B_1 \cap B_2}] = [A_{B_1} \cap A_{B_2}] ,$$
$$[A_{B_1 \cup B_2 \cup \dots}] = [A_{B_1} \cup A_{B_2} \cup \dots] .$$

In order to restore $[f]$, use a countable base of Y...

Hints to exercises

7a2: Consider $m\{x : \inf_k f_{n+k}(x) \ge c\}$.
7b3: $\sum_k 3^{-n-k} \le \frac{1}{2} \cdot 3^{-n}$.
7b4: use 7b2–7b3.
7c1: $f(\sum_n 2^{-n}\beta_n) = \sum_n 3^{-n} \cdot 2\beta_n$ is continuous quasi everywhere.
7c3: apply 7c2(c) to $\rho(x, f(\cdot)) = \lim_n \rho(x, f_n(\cdot))$.

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