10 Typical compact sets

10a Covering, packing, volume, and dimension

Covering numbers $\mathcal{N}_\varepsilon(X)$ and packing numbers $\mathcal{M}_\varepsilon(X)$ (natural numbers or ∞) are defined for $\varepsilon > 0$ and a metric space X by[1]

$$\mathcal{N}_\varepsilon(X) = \inf\{|A| : \forall x \in X \ \exists a \in A \ \ \rho(x,a) < \varepsilon\} ,$$
$$\mathcal{M}_\varepsilon(X) = \sup\{|B| : \forall b_1, b_2 \in B \ \ (b_1 \ne b_2 \implies \rho(b_1,b_2) \ge \varepsilon)\} ;$$

here A, B run over all finite subsets of X, and $|\dots|$ is the number of elements.

10a1 Lemma. $\mathcal{M}_{2\varepsilon}(X) \le \mathcal{N}_\varepsilon(X) \le \mathcal{M}_\varepsilon(X)$.

Proof. $\mathcal{M}_{2\varepsilon}(X) \le \mathcal{N}_\varepsilon(X)$: we have a one-to-one map $B \to A$, since $\rho(b_1,a_1) < \varepsilon$ and $\rho(b_2,a_2) < \varepsilon$ and $b_1 \ne b_2$ imply $a_1 \ne a_2$.

$\mathcal{N}_\varepsilon(X) \le \mathcal{M}_\varepsilon(X)$: if $\mathcal{M}_\varepsilon(X) < \infty$, we take a *maximal* B and note that it is a possible A. $\square$

10a2 Exercise. The following three conditions on a metric space X are equivalent:

(a) $\forall \varepsilon > 0 \ \ \mathcal{N}_\varepsilon(X) < \infty$;
(b) $\forall \varepsilon > 0 \ \ \mathcal{M}_\varepsilon(X) < \infty$;
(c) every sequence (of points of X) has a Cauchy subsequence.

Prove it.

10a3 Corollary. The following three conditions on a *complete* metric space X are equivalent:

(a) $\forall \varepsilon > 0 \ \ \mathcal{N}_\varepsilon(X) < \infty$;
(b) $\forall \varepsilon > 0 \ \ \mathcal{M}_\varepsilon(X) < \infty$;
(c) X is compact.

[1] Not equivalently, but equally well, one may use $\rho(x,a) \le \varepsilon$ in concert with $\rho(b_1,b_2) > \varepsilon$.

Interestingly, it appears that (a) and (b) do not depend on the choice of a *complete* metric on a (completely) metrizable space. (However, $(0,1)$ is homeomorphic to $\mathbb{R}$...)

A subset E of a metric space X is itself a metric space; thus, $\mathcal{N}_\varepsilon(E)$, $\mathcal{M}_\varepsilon(E)$ are well-defined, and 10a1–10a3 apply. Compact subsets of $\mathbb{R}^n$ are a notable special case. For a cube $E \subset \mathbb{R}^n$ it is easy to see that $\mathcal{N}_\varepsilon(E) \le C/\varepsilon^n$ and $\mathcal{M}_\varepsilon(E) \ge c/\varepsilon^n$ for some $c, C \in (0,\infty)$ (dependent on n but not E). Using the inequality $\mathcal{M}_{2\varepsilon}(E) \le \mathcal{N}_\varepsilon(E)$ we get

$$\mathcal{N}_\varepsilon(E) \asymp \mathcal{M}_\varepsilon(E) \asymp \frac{1}{\varepsilon^n} \,;$$

here $\alpha \asymp \beta$ means that $c\alpha \le \beta \le C\alpha$ for some $c, C \in (0,\infty)$. The same holds for every bounded set $E \subset \mathbb{R}^n$ with nonempty interior (in particular, a ball). For such E we get

$$\frac{\log \mathcal{N}_\varepsilon(E)}{\log 1/\varepsilon} \to n\,, \quad \frac{\log \mathcal{M}_\varepsilon(E)}{\log 1/\varepsilon} \to n \quad \text{as } \varepsilon \to 0+ \,.$$

Accordingly, one defines the lower and upper Minkowski(-Bouligand) dimension[1]

$$\underline{\dim}_{\mathrm{M}}(E) = \liminf_{\varepsilon\to 0+} \frac{\log \mathcal{N}_\varepsilon(E)}{\log 1/\varepsilon}\,, \quad \overline{\dim}_{\mathrm{M}}(E) = \limsup_{\varepsilon\to 0+} \frac{\log \mathcal{N}_\varepsilon(E)}{\log 1/\varepsilon}\,,$$

and if these are equal, the Minkowski dimension $\dim_{\mathrm{M}}(E)$ is equal to both.

(Equivalently, $\mathcal{M}_\varepsilon(E)$ may be used.)

Now we turn to a bounded set $E \subset \mathbb{R}^n$ and Lebesgue measure of its closed ε-neighborhood $E_{+\varepsilon}$.

10a4 Lemma. For all $\varepsilon > 0$,

$$\mathcal{M}_{2\varepsilon}(E) \le \frac{m(E_{+\varepsilon})}{C_n \varepsilon^n} \le 2^n \mathcal{N}_\varepsilon(E)\,,$$

where C_n is the volume of the n-dimensional unit ball.

Proof. First, $\mathcal{N}_\varepsilon(E) = |A|$, $E \subset A_{+\varepsilon}$, thus $E_{+\varepsilon} \subset A_{+2\varepsilon}$ and $m(E_{+\varepsilon}) \le C_n(2\varepsilon)^n|A|$.

Second, $\mathcal{M}_{2\varepsilon}(E) = |B|$, $B \subset E$, thus $m(E_{+\varepsilon}) \ge C_n\varepsilon^n|B|$. □

[1] Also "box (counting) dimension", and (for the upper dimension) "entropy dimension", "Kolmogorov dimension", "Kolmogorov capacity". By the way, the well-known Hausdorff dimension never exceeds the lower Minkowski dimension.

10a5 Corollary. If $\dim_{\mathrm{M}}(E)$ exists, then

$$\frac{\log m(E_{+\varepsilon})}{\log \varepsilon} \to n - \dim_{\mathrm{M}}(E) \quad \text{as } \varepsilon \to 0+ \, ;$$

and in every case,

$$n - \overline{\dim}_{\mathrm{M}}(E) = \liminf_{\varepsilon\to 0+} \frac{\log m(E_{+\varepsilon})}{\log \varepsilon} \le \limsup_{\varepsilon\to 0+} \frac{\log m(E_{+\varepsilon})}{\log \varepsilon} = n - \underline{\dim}_{\mathrm{M}}(E) \, .$$

10b A space of compact sets

Given a metric space X, we denote by $\mathbf{K}(X)$ the set of all nonempty compact subsets of X. For each $K \in \mathbf{K}(X)$ we introduce its distance function $d_K : X \to [0,\infty)$ by

$$d_K(x) = \operatorname{dist}(x, K) = \min_{y\in K} \rho(x,y) \, .$$

Note that $|d_K(x) - d_K(y)| \le \rho(x,y)$; $K = \{x : d_K(x) = 0\}$; and $K_1 \subset K_2 \iff d_{K_1} \ge d_{K_2}$. Also, $d_{K_1\cup K_2} = \min(d_{K_1}, d_{K_2})$, while the evident inequality $d_{K_1\cap K_2} \ge \max(d_{K_1}, d_{K_2})$ is generally strict (even if $K_1 \cap K_2 \ne \emptyset$).

We endow $\mathbf{K}(X)$ with the *Hausdorff metric* d_{H},

$$d_{\mathrm{H}}(K_1, K_2) = \|d_{K_1} - d_{K_2}\| = \sup_{x\in X} |d_{K_1}(x) - d_{K_2}(x)| \, .$$

10b1 Exercise. (a) $\sup_{x\in X}\big(d_{K_1}(x) - d_{K_2}(x)\big) = \max_{x\in K_2} d_{K_1}(x)$;

(b) $d_{\mathrm{H}}(K_1,K_2) = \max\big(\max_{x\in K_2} d_{K_1}(x), \max_{x\in K_1} d_{K_2}(x)\big)$.

Prove it.

Denoting $K_{+\varepsilon} = \{x : d_K(x) \le \varepsilon\}$ we have

$$d_{\mathrm{H}}(K_1,K_2) = \min\{\varepsilon : K_1 \subset (K_2)_{+\varepsilon} \wedge K_2 \subset (K_1)_{+\varepsilon}\} \, .$$

Here is a metric-free description[1] of the topology on $\mathbf{K}(X)$.

10b2 Exercise. The following two conditions on $K, K_1, K_2, \dots \in \mathbf{K}(X)$ are equivalent:

(a) $K_n \to K$;

(b) for every open $U \subset X$,

(b1) if $K \subset U$ then $K_n \subset U$ for all n large enough;

(b2) if $K \cap U \ne \emptyset$ then $K_n \cap U \ne \emptyset$ for all n large enough.

Prove it.

[1] So-called Vietoris topology on $\mathbf{K}(X)$.

If two metrics on X are equivalent then the two corresponding Hausdorff metrics on $\mathbf{K}(X)$ are equivalent. Thus, a *metrizable* space $\mathbf{K}(X)$ is well-defined for every *metrizable* space X.

10b3 Exercise. If X is separable then $\mathbf{K}(X)$ is separable.
Prove it.

We return to a metric space X.

10b4 Exercise. Let $K_1, K_2, \dots \in \mathbf{K}(X)$, $K_1 \supset K_2 \supset \dots$, then $d_{K_n} \uparrow d_K$ where $K = \cap_n K_n$.
Prove it.

10b5 Exercise. Let $K_1, K_2, \dots \in \mathbf{K}(X)$ be such that the set $K = \mathrm{Cl}(K_1 \cup K_2 \cup \dots)$ is compact; then
(a) $d_K = \inf_n d_{K_n}$;
(b) $\liminf_n d_{K_n} = d_{K_\infty}$ where $K_\infty = \cap_n \mathrm{Cl}(K_n \cup K_{n+1} \cup \dots)$ is the so-called topological upper limit of K_n;
(c) $K_\infty \supset \limsup_n K_n$, and this inequality is generally strict.
Prove it.

10b6 Exercise. Let $K_1, K_2, \dots \in \mathbf{K}(X)$ and $\varepsilon_n \to 0$. If X is complete then the set

$$\bigcap_n (K_n)_{+\varepsilon_n}$$

is compact.
Prove it.

10b7 Proposition. If X is complete then $\mathbf{K}(X)$ is complete.

Proof. Given a Cauchy sequence $K_1, K_2, \dots \in \mathbf{K}(X)$, we take $\varepsilon_n \to 0$ such that $d_{\mathrm{H}}(K_n, K_{n+k}) \le \varepsilon_n$, then $K_{n+k} \subset (K_n)_{+\varepsilon_n}$, therefore

$$\bigcup_n K_n \subset \bigcap_n (K_1 \cup \dots \cup K_n)_{+\varepsilon_n} .$$

The latter is compact by 10b6. Thus, $\mathrm{Cl}(\cup_n K_n)$ is compact. By 10b5, $\liminf_n d_{K_n} = d_{K_\infty}$. However, d_{K_n} are a Cauchy sequence; thus $\|d_{K_n} - d_{K_\infty}\| \to 0$, that is, $K_n \to K_\infty$. $\square$

10b8 Corollary. (a) If X is completely metrizable then $\mathbf{K}(X)$ is completely metrizable;
(b) if X is Polish then $\mathbf{K}(X)$ is Polish.

10c Dimensions of typical sets

10c1 Theorem. [1] Quasi all $K \in \mathbf{K}(\mathbb{R}^n)$ satisfy

$$\underline{\dim}_{\mathrm{M}}(K) = 0 \, , \quad \overline{\dim}_{\mathrm{M}}(K) = n \, .$$

This fact should be another manifestation of the phenomenon seen in 1e1(b). I wonder, is there a general theorem that implies both as special cases?

10c2 Corollary. Quasi all $K \in \mathbf{K}(\mathbb{R}^n)$ are null sets, and (therefore) nowhere dense.

Strangely, small sets are the majority…

In order to avoid the question of continuity of $m(K_{+\varepsilon})$ in K (for a fixed ε) we introduce

$$f_\varepsilon(K) = \frac{1}{\varepsilon} \int_0^\varepsilon m(K_{+a}) \, \mathrm{d}a = \int_{\mathbb{R}^n} \Big(1 - \frac{1}{\varepsilon} d_K \Big)^+ \, \mathrm{d}m \, .$$

If $K_m \to K$ then $d_{K_m} \to d_K$ uniformly, thus $f_\varepsilon(K_m) \to f_\varepsilon(K)$ (since the relevant part of $\mathbb{R}^n$ is bounded). It means that $f_\varepsilon : \mathbf{K}(\mathbb{R}^n) \to (0, \infty)$ is continuous.

By monotonicity,

$$\frac{1}{2} m(K_{+\varepsilon/2}) \le \frac{1}{\varepsilon} \int_0^\varepsilon m(K_{+a}) \, \mathrm{d}a \le m(K_{+\varepsilon}) \, ,$$

therefore

$$\bigg(1 + O\Big(\frac{1}{\log 1/\varepsilon} \Big) \bigg) \frac{\log m(K_{+\varepsilon/2})}{\log(\varepsilon/2)} \le \frac{\log f_\varepsilon(K)}{\log \varepsilon} \le \frac{\log m(K_{+\varepsilon})}{\log \varepsilon} \, .$$

In combination with 10a5 it gives

$$\frac{\log f_\varepsilon(K)}{\log \varepsilon} \to n - \dim_{\mathrm{M}}(K) \quad \text{as } \varepsilon \to 0+$$

if $\dim_{\mathrm{M}}(K)$ exists; and in every case,

(10c3)
$$n - \overline{\dim}_{\mathrm{M}}(K) = \liminf_{\varepsilon \to 0+} \frac{\log f_\varepsilon(K)}{\log \varepsilon} \le \limsup_{\varepsilon \to 0+} \frac{\log f_\varepsilon(K)}{\log \varepsilon} = n - \underline{\dim}_{\mathrm{M}}(K) \, .$$

[1] P.M. Gruber (1989) "Dimension and structure of typical compact sets, continua and curves", Monatshefte für Mathematik **108**, 149–164.

Proof of Theorem 10c1. Finite K are dense in $\mathbf{K}(\mathbb{R}^n)$ and are of Minkowski dimension 0. Thus,

$$\frac{\log f_\varepsilon(K)}{\log \varepsilon} \to n \quad (\text{as } \varepsilon \to 0)$$

on a dense set. By 5b9,

$$\limsup_{\varepsilon\to 0+} \frac{\log f_\varepsilon(K)}{\log \varepsilon} \ge n \quad (\text{thus, } = n)$$

for quasi all $K \in \mathbf{K}(\mathbb{R}^n)$. By (10c3), $\underline{\dim}_{\mathrm{M}}(K) = 0$ for quasi all K. On the other hand, sets of Minkowski dimension n are also dense (try finite unions of balls; or even a ball plus a finite set). Thus,

$$\frac{\log f_\varepsilon(K)}{\log \varepsilon} \to 0 \quad (\text{as } \varepsilon \to 0+)$$

on a dense set. By 5b9 (again),

$$\liminf_{\varepsilon\to 0+} \frac{\log f_\varepsilon(K)}{\log \varepsilon} \le 0 \quad (\text{thus, } = 0)$$

for quasi all $K \in \mathbf{K}(\mathbb{R}^n)$. By (10c3), $\overline{\dim}_{\mathrm{M}}(K) = n$ for quasi all K. □

10c4 Exercise. Quasi all $K \in \mathbf{K}(\mathbb{R}^n)$ satisfy

$$\forall U \ \big(K \cap U \ne \emptyset \implies \overline{\dim}_{\mathrm{M}}(K \cap U) = n \big) ,$$

where U runs over all open sets.

Prove it.

10c5 Corollary. Quasi all $K \in \mathbf{K}(\mathbb{R}^n)$ are perfect sets.

By Theorem 10c1, quasi all $K \in \mathbf{K}(\mathbb{R}^n)$ satisfy

$$\forall \alpha > 0 \quad \liminf_{\varepsilon\to 0+} \frac{m(K_{+\varepsilon})}{\varepsilon^{n-\alpha}} = 0 , \quad \limsup_{\varepsilon\to 0+} \frac{m(K_{+\varepsilon})}{\varepsilon^{\alpha}} = \infty .$$

On the other hand,

$$m(K_{+\varepsilon}) \to 0 , \quad \frac{m(K_{+\varepsilon})}{\varepsilon^n} \to \infty \quad (\text{as } \varepsilon \to 0+) \tag{10c6}$$

since $m(K) = 0$, and K is infinite (and $\frac{m(E_{+\varepsilon})}{\varepsilon^n} \to C_n |E|$ for finite E).

A more detailed analysis leads to a stronger result.

We recall the "anti-Egorov" phenomenon discussed in Sect. 7a (recall also 9b2) and give it a name.[1]

[1]Not a standard terminology.

10c7 Definition. Let X be a completely metrizable space, and $f_1, f_2, \dots : X \to (0, \infty)$. We say that quasi everywhere

$$f_n \to 0 \text{ with nothing to spare}$$

if $\{x : f_n(x) \to 0\}$ is comeager, but $\{x : A_n f_n(x) \to 0\}$ is meager whenever $A_n \to \infty$.

Similarly, we say that quasi everywhere

$$f_n \to \infty \text{ with nothing to spare}$$

if $\{x : f_n(x) \to \infty\}$ is comeager, but $\{x : a_n f_n(x) \to \infty\}$ is meager whenever $a_n \to 0$.

(And the same for $\varepsilon \to 0+$ instead of $n \to \infty$).

10c8 Theorem. For quasi all $K \in \mathbf{K}(\mathbb{R}^n)$,

$$m(K_{+\varepsilon}) \to 0 \text{ (as } \varepsilon \to 0+\text{) with nothing to spare,}$$
$$\frac{m(K_{+\varepsilon})}{\varepsilon^n} \to \infty \text{ (as } \varepsilon \to 0+\text{) with nothing to spare.}$$

Proof. Convergence is already established, see (10c6); "nothing to spare" will be proved. Using 10a4 (and 10a1) we reformulate it equivalently as follows:

$$\mathcal{N}_\varepsilon(K) \to \infty \text{ (as } \varepsilon \to 0+\text{) with nothing to spare,}$$
$$\varepsilon^n \mathcal{N}_\varepsilon(K) \to 0 \text{ (as } \varepsilon \to 0+\text{) with nothing to spare.}$$

The first relation. Let $a(\varepsilon) \to 0$; we have to prove that $\{K : a(\varepsilon)\mathcal{N}_\varepsilon(K) \to \infty\}$ is meager. It is sufficient to prove for arbitrary ε_0 that the set $S = \{K : \forall \varepsilon \le \varepsilon_0 \;\; a(\varepsilon)\mathcal{N}_\varepsilon(K) \ge 1\}$ is nowhere dense. We'll prove that a finite $K_0 \in \mathbf{K}(\mathbb{R}^n)$ cannot belong to the closure of S; this is sufficient, since these K_0 are dense in $\mathbf{K}(\mathbb{R}^n)$.

We take $\varepsilon \le \varepsilon_0$ such that $a(\varepsilon)|K_0| < 1$. Every K such that $d_{\mathrm{H}}(K_0, K) \le \varepsilon$ satisfies $\mathcal{N}_\varepsilon(K) \le |K_0|$, thus, $a(\varepsilon)\mathcal{N}_\varepsilon(K) < 1$. We see that S misses the ε-neighborhood of K_0.

The second relation. Let $A(\varepsilon) \to \infty$; we have to prove that $\{K : A(\varepsilon)\varepsilon^n\mathcal{N}_\varepsilon(K) \to 0\}$ is meager. It is sufficient to prove for arbitrary ε_0 that the set $S = \{K : \forall \varepsilon \le \varepsilon_0 \;\; A(\varepsilon)\varepsilon^n\mathcal{N}_\varepsilon(K) \le 1\}$ is nowhere dense. We'll prove that $K_0 \in \mathbf{K}(\mathbb{R}^n)$ with nonempty interior cannot belong to the closure of S; this is sufficient, since these K_0 are dense in $\mathbf{K}(\mathbb{R}^n)$.

There exists $c > 0$ such that $\varepsilon^n\mathcal{N}_\varepsilon(K_0) \ge c$ for all ε. We take $\varepsilon \le \varepsilon_0$ such that $A(\varepsilon)c > 2^n$. Every K such that $d_{\mathrm{H}}(K_0, K) \le \varepsilon$ satisfies $\mathcal{N}_\varepsilon(K) \ge \mathcal{N}_{2\varepsilon}(K_0)$, thus, $A(\varepsilon)\varepsilon^n\mathcal{N}_\varepsilon(K) > 1$. We see that S misses the ε-neighborhood of K_0. □

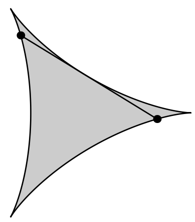
In this sense,

$$\varepsilon^n \ll m(K_{+\varepsilon}) \ll 1 \quad \text{and} \quad 1 \ll \mathcal{N}_\varepsilon(K) \ll \frac{1}{\varepsilon^n}$$

(as $\varepsilon \to +0$) with nothing to spare.

10d Besicovitch sets

By Besicovitch sets we mean planar compact sets that contain unit line segments in every direction. We want to minimize the area of such set. It was conjectured that the *deltoid* is optimal, of area $\pi/8$. Amazingly, the minimal area is zero![1]



Deltoid. Its boundary is $\{\frac{1}{2}e^{it} + \frac{1}{4}e^{-2it} : 0 \le t \le 2\pi\}$.

10d1 Exercise. The set B of all Besicovitch sets is a closed subset of $\mathbf{K}(\mathbb{R}^2)$.
Prove it.

Thus we may talk about typical Besicovitch sets.

10d2 Theorem. [2] Quasi all Besicovitch sets are null sets.

A spectacular manifestation of the tendency "small sets are the majority"!

On $\mathbf{K}(\mathbb{R}^n)$, Lebesgue measure $K \mapsto m(K)$ is an upper semicontinuous function. Moreover, $K \mapsto \mu(K)$ is upper semicontinuous on $\mathbf{K}(X)$ for every locally finite measure μ on X. Proof: $K_{+\varepsilon} \downarrow K$, therefore $\mu(K_{+\varepsilon}) \downarrow \mu(K)$; if $\mu(K) < a$ then $\mu(K_{+\varepsilon}) < a$ for a small ε, and $\mu(K_1) < a$ whenever $d_{\mathrm{H}}(K_1, K) \le \varepsilon$.

Thus, in order to prove Theorem 10d2 it is sufficient to prove that $\{K \in B : m(K) < \varepsilon\}$ is dense in B for all ε.

We divide the set of all directions in two subsets: these closer to the x axis, and to y axis. We have $B = B_1 \cap B_2$ where B_1 consists of sets that contain unit line segment in every direction closer to the x axis, and B_2 to y. It is sufficient to prove that $\{K \in B_1 : m(K) < \varepsilon\}$ is dense in B_1, since if $K_1 \in B_1$, $K_2 \in B_2$, $m(K_1) < \varepsilon$, $m(K_2) < \varepsilon$, then $K_1 \cup K_2 \in B$, $m(K_1 \cup K_2) < 2\varepsilon$ and $d_{\mathrm{H}}(K_1 \cup K_2, K) \le \max\big(d_{\mathrm{H}}(K_1, K), d_{\mathrm{H}}(K_2, K)\big)$ (think, why).

We'll prove a stronger claim: quasi all sets of B_1 are null sets.

[1] Some authors define Besicovitch sets as null sets with that property.

[2] T.W. Körner (2003) "Besicovitch via Baire", Studia Math. **158**, 65–78. See also Sect. 4.5 in book: E.M. Stein and R. Shakarchi, "Functional analysis", Princeton 2011.

To this end it is sufficient to prove that

$$m\big(K \cap ([a, a+\varepsilon] \times \mathbb{R})\big) < A\varepsilon^2 \tag{10d3}$$

for quasi all $K \in B_1$, whenever $a \in \mathbb{R}$, $\varepsilon > 0$; here A is some absolute constant.

Given a and ε, we have an open set of such $K \in B_1$, and we'll prove that this open set is dense in B_1. Given $K \in B_1$ and N, we seek $K_1 \in B_1$ satisfying (10d3) and $d_{\mathrm{H}}(K_1, K) = O(1/N)$.[1]

We take directions $d_1, \dots, d_N$, closer to the x axis, that are $O(1/N)$-dense among all such directions. For each $i = 1, \dots, N$ we choose a unit line segment $S_i \subset K$ in the direction d_k. Rotating S_i by angles $\pm O(1/N)$ around one of its points (specified below) we get $\tilde{S}_i \in \mathbf{K}(\mathbb{R}^2)$ such that the set $\tilde{S} = \tilde{S}_1 \cap \dots \cup \tilde{S}_N$ belongs to B_1 and $\tilde{S} \subset K_{+O(1/N)}$.

We choose the center of the rotation as the point of S_i most close to the line $\{a + \frac{\varepsilon}{2}\} \times \mathbb{R}$ (be it on the line or not). Then $m(\tilde{S}_i) = O(\varepsilon^2/N)$ (think, why), thus $m(\tilde{S}) = O(\varepsilon^2)$. It remains to take $K_1 = \tilde{S} \cup K_0$ where $K_0 \subset K$ is a compact null set (even finite, if you like) such that $K \subset A_{+O(1/N)}$; indeed, then $K \subset (K_1)_{+O(1/N)}$ and $K_1 \subset K_{+O(1/N)}$.

Hints to exercises

10a2: (c)$\Longrightarrow$(a): if no A is finite then some B is infinite; (a)$\Longrightarrow$(c): if A is finite then some ε-ball contains x_n for infinitely many n.

10b3: try finite (compact) sets.

10b6: use 10a3.

10c4: use a countable basis.

10d1: $K_n \to K_\infty$.

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[1]All $O(1/N)$ are absolute (I mean, with absolute constants).