# Foreword

## A problem, for example

Here is a motivating example. Given a sequence of random variables $\zeta_1, \zeta_2, \dots : \Omega \to \mathbb{R}$ and continuous functions $f_1, f_2, \dots : \mathbb{R}^3 \to \mathbb{R}$, we consider a random function

$$\psi_\omega : \mathbb{R}^3 \to \mathbb{R}\,, \quad \psi_\omega = \sum_{n=1}^{\infty} \zeta_n(\omega) f_n(x) \quad \text{for } x \in \mathbb{R}^3,\, \omega \in \Omega\,,$$

assuming that the series converges a.s. to a continuous function. We get a random field on $\mathbb{R}^3$. We consider the random closed set

$$Z_\omega = \{x \in \mathbb{R}^3 : \psi_\omega(x) = 0\} \quad \text{for } \omega \in \Omega$$

and its connected components. We may ask many question, such as

* are the components bounded?
* what about the number of components in a large ball?
* what about their topological properties?

And so on. Surely, the answers depend on the properties of the random field. In order to answer such a question for a given random field, one usually needs ingenuity rather than a general theory. However, are these questions well-defined? Are we sure that the relevant subsets of $\Omega$ are measurable? Here the general theory should help.

Random variables $\zeta_n$ are real-valued; this is quite simple. The random field $(\psi_\omega)_\omega$ is a random element of a space of continuous functions $\mathbb{R}^3 \to \mathbb{R}$. This space is infinite-dimensional, but still, not unusual; probably one can work in an appropriate Hilbert or Banach space. The random set $(Z_\omega)_\omega$ is a random element of the set (space?) of closed subsets of $\mathbb{R}^3$. Quite nonlinear! Is it a tractable space? Of which kind? But wait, we need the set of connected components of $Z_\omega$. This is a random element of the set (space??) of (nice, or not??) subsets of the previous "quite nonlinear" space(?). Is *this* tractable??

You might expect one of the three "discouraging" answers:

* yes, all that is tractable easily; just learn some relevant definitions and their straightforward implications;
* no, all that is generally intractable; nonmeasurable sets can appear easily; try to prove measurability in every needed special case, separately and specifically;
* well, these are fine points of the set theory; the answers can be "yes" or "no" depending on additional axioms; try to prove measurability in every needed case specifically.

The true answer is less expected and more encouraging:

* yes, *most cases* are tractable, but not easily; the needed theory is quite nontrivial, but not overcomplicated (you do not need even the transfinite induction). However, *some cases* are indeed intractable; try to prove measurability in every such case specifically.

## A result, for example

Note that a subset of $\mathbb{R}$ is a Borel set if and only if it belongs to the least set (of sets) satisfying the following conditions:

* every interval is a Borel set;
* the complement of a Borel set is a Borel set;
* the union of an infinite sequence of Borel sets is a Borel set.

In the same spirit, given a probability space[1] $(\Omega, \mathcal{F}, P)$, we define a *random Borel set* as a map $X$ from $\Omega$ to the set of all subsets of $\mathbb{R}$ that belongs to the least set (of maps) satisfying the following conditions:

* if $A \in \mathcal{F}$ and $I \subset \mathbb{R}$ is an interval then the map $\omega \mapsto \begin{cases} I & \text{for } \omega \in A, \\ \emptyset & \text{otherwise} \end{cases}$ is a random Borel set;
* if $X$ is a random Borel set then the map $\omega \mapsto \mathbb{R} \setminus X(\omega)$ is a random Borel set;
* if $X_1, X_2, \dots$ are random Borel sets then the map $\omega \mapsto X_1(\omega) \cup X_2(\omega) \cup \dots$ is a random Borel set.

One of the most basic questions about a random Borel set is: what is the probability that it is empty? That is, $P\big(\{\omega \in \Omega : X(\omega) = \emptyset\}\big) = ?$ But wait; are you sure that $\{\omega : X(\omega) = \emptyset\} \in \mathcal{F}$? It is easy to see that $\{\omega : x \in X(\omega)\} \in \mathcal{F}$ for every $x \in \mathbb{R}$; but we need the union of these sets over all $x \in \mathbb{R}$, — uncountably many...

**Fact.** It may happen that $\{\omega : X(\omega) = \emptyset\} \notin \mathcal{F}$. Moreover, this may happen when $\Omega = [0, 1]$ and $\mathcal{F}$ is the Borel $\sigma$-algebra on $[0, 1]$.

**Fact.** The set $\{\omega : X(\omega) = \emptyset\}$ is $P$-measurable; that is, there exist $A, B \in \mathcal{F}$ such that $A \subset \{\omega : X(\omega) = \emptyset\} \subset B$ and $P(A) = P(B)$.

The latter fact shows that the probability that a random Borel set is empty is well-defined. The former fact shows that the proof cannot be simple.

[1] Just a measure space such that $P(\Omega) = 1$.

## Why this name, "measurability and continuity"

Relations between measurability and continuity may seem to be evident, but they are not. The same can be said about relations between $\sigma$-algebras and topologies. Evidently,

* continuous functions are measurable, but measurable functions are generally discontinuous;
* a $\sigma$-algebra is often introduced using a preexisting topology, but the topology cannot be restored from the $\sigma$-algebra.

Surprisingly,

* in many cases a $\sigma$-algebra can be introduced and used irrespective of any topology, and is more inherent than a topology;
* every measurable function is continuous in some useful topology (dependent on the function);
* in many cases, deep results about $\sigma$-algebras are proved using an auxiliary topology (constructed rather than preexisting).