1 Basic notions and constructions

Measures (and probabilities) aside, in this section we concentrate on sigma-algebras and other systems of sets.

1a Algebra of sets

Algebra of sets is an easy matter. Algebra generated by given sets is described explicitly. Closed-and-open subsets of the Cantor set are an algebra instrumental in understanding the general case.

Let X be a set, $2^X = \{A : A \subset X\}$ the set of all subsets of X (including X itself). For arbitrary $\mathcal{E} \subset 2^X$ we denote

$$\begin{gathered}\sim\!\mathcal{E} = \{X \setminus A : A \in \mathcal{E}\}\,,\\ \mathcal{E}_{\mathrm{d}} = \{A_1 \cap \dots \cap A_n : A_1, \dots, A_n \in \mathcal{E}, n = 0, 1, 2, \dots\}\,,\\ \mathcal{E}_{\mathrm{s}} = \{A_1 \cup \dots \cup A_n : A_1, \dots, A_n \in \mathcal{E}, n = 0, 1, 2, \dots\}\,.\end{gathered}$$

(For $n = 0$ the union is $\emptyset$ and the intersection is X.) Clearly, $\mathcal{E}_{\mathrm{d}} \supset \mathcal{E}$ and $\mathcal{E}_{\mathrm{s}} \supset \mathcal{E}$. Also, $\mathcal{E}_{\mathrm{dd}} = \mathcal{E}_{\mathrm{d}}$ (here $\mathcal{E}_{\mathrm{dd}} = (\mathcal{E}_{\mathrm{d}})_{\mathrm{d}}$) and $\mathcal{E}_{\mathrm{ss}} = \mathcal{E}_{\mathrm{s}}$. If $\mathcal{E}$ is finite then $\mathcal{E}_{\mathrm{d}}, \mathcal{E}_{\mathrm{s}}$ are finite. If $\mathcal{E}$ is countable then $\mathcal{E}_{\mathrm{d}}, \mathcal{E}_{\mathrm{s}}$ are countable.

1a1 Core exercise. Prove that $\sim\sim\mathcal{E} = \mathcal{E}$; $(\sim\mathcal{E})_{\mathrm{d}} = \sim(\mathcal{E}_{\mathrm{s}})$; $(\sim\mathcal{E})_{\mathrm{s}} = \sim(\mathcal{E}_{\mathrm{d}})$.

1a2 Example. $X = \{0,1\}^n$, $\mathcal{E} = \{A_1, \dots, A_n\}$ where

$$A_k = \{x \in X : x(k) = 1\} = \{0,1\}^{k-1} \times \{1\} \times \{0,1\}^{n-k} \quad \text{for } k = 1, \dots, n\,.$$

Or equivalently, $X = 2^{\{1,\dots,n\}}$, $A_k = \{x \in X : k \in x\}$.

1a3 Core exercise. Let $X, \mathcal{E}$ be as in 1a2. Prove that $\mathcal{E}_{\mathrm{d}}$ contains exactly 2^n sets.

1a4 Extra exercise. Let $X, \mathcal{E}$ be as in 1a2. For arbitrary $A \subset X$ prove that $A \in \mathcal{E}_{\mathrm{ds}}$ if and only if $\forall x, y \in X \; (x \le y \wedge x \in A \implies y \in A)$.

1a5 Extra exercise. Let $X, \mathcal{E}$ be as in 1a2. Prove that $(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{d}}$ contains exactly $3^n + 1$ sets.

1a6 Core exercise. Let $X, \mathcal{E}$ be as in 1a2. Prove that $(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{ds}} = 2^X$.

Example 1a2 is quite special, but instrumental in understanding the general case, as we will see soon.

Given sets X, Y and a map $\varphi : X \to Y$ (generally not invertible), we have the "inverse image" ("pullback") map $\Phi = \varphi^{-1} : 2^Y \to 2^X$, $\Phi(B) = \varphi^{-1}(B) = \{x : \varphi(x) \in B\}$. Further, given $\mathcal{F} \subset 2^Y$, we get $\Phi(\mathcal{F}) \subset 2^X$, $\Phi(\mathcal{F}) = \{\Phi(B) : B \in \mathcal{F}\} = \{\varphi^{-1}(B) : B \in \mathcal{F}\}$. On the other hand, given $\mathcal{E} \subset 2^X$, we get $\Phi^{-1}(\mathcal{E}) \subset 2^Y$, $\Phi^{-1}(\mathcal{E}) = \{B \subset Y : \Phi(B) \in \mathcal{E}\} = \{B \subset Y : \varphi^{-1}(B) \in \mathcal{E}\}$.

1a7 Core exercise. Prove that[1]

$$\Phi({\sim}\mathcal{F}) = {\sim}(\Phi(\mathcal{F}))\,, \quad \Phi(\mathcal{F}_{\mathrm{d}}) = (\Phi(\mathcal{F}))_{\mathrm{d}}\,, \quad \Phi(\mathcal{F}_{\mathrm{s}}) = (\Phi(\mathcal{F}))_{\mathrm{s}}$$

whenever $\varphi : X \to Y$ and $\mathcal{F} \subset 2^Y$. (As before, $\Phi = \varphi^{-1} : 2^Y \to 2^X$.)

Given a finite $\mathcal{E} \subset 2^X$ and its enumeration $\mathcal{E} = \{A_1, \dots, A_n\}$, we introduce a map $\varphi : X \to \{0,1\}^n$ by

$$\varphi(x) = \big(\mathbf{1}_{A_1}(x), \dots, \mathbf{1}_{A_n}(x)\big)\,; \tag{1a8}$$

here $\mathbf{1}_A(x) = 1$ for $x \in A$ and 0 for $x \in X \setminus A$. Or, if you like, we may avoid enumeration of $\mathcal{E}$ as follows: $\varphi : X \to \{0,1\}^{\mathcal{E}}$, $\varphi(x)(A) = \mathbf{1}_A(x)$ for $x \in X$, $A \in \mathcal{E}$. Or equivalently, $\varphi : X \to 2^{\mathcal{E}}$, $\varphi(x) = \{A \in \mathcal{E} : x \in A\}$.

Now let us denote $X, \mathcal{E}$ of 1a2 by $X_n, \mathcal{E}_n$, releasing $X, \mathcal{E}$ for the general case. The map $\varphi : X \to X_n$ introduced above gives

$$\mathcal{E} = \Phi(\mathcal{E}_n)$$

(think, why). By 1a7 (applied several times),

$$(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{ds}} = \Phi\big((\mathcal{E}_n \cup {\sim}\mathcal{E}_n)_{\mathrm{ds}}\big)\,;$$

by 1a6, $(\mathcal{E}_n \cup {\sim}\mathcal{E}_n)_{\mathrm{ds}} = 2^{X_n}$; thus,

$$(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{ds}} = \Phi\big(2^{X_n}\big)\,. \tag{1a9}$$

(As before, $\Phi = \varphi^{-1} : 2^Y \to 2^X$.)

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[1] Likewise, if $f : G_1 \to G_2$ is a homomorphism of groups then $f(A \cdot A) = f(A) \cdot f(A)$ for every $A \subset G_1$.

1a10 Definition. A set $\mathcal{E} \subset 2^X$ is an *algebra*[1] (of sets) (on X) if $X \setminus A$, $A \cap B$ and $A \cup B$ belong to $\mathcal{E}$ for all $A, B \in \mathcal{E}$. In other words, if

$$\sim\!\mathcal{E} \subset \mathcal{E}\,, \quad \mathcal{E}_{\mathrm{d}} \subset \mathcal{E}\,, \quad \mathcal{E}_{\mathrm{s}} \subset \mathcal{E}\,.$$

(By 1a1, the conditions $\mathcal{E}_{\mathrm{d}} \subset \mathcal{E}$, $\mathcal{E}_{\mathrm{s}} \subset \mathcal{E}$ are equivalent, given $\sim\!\mathcal{E} \subset \mathcal{E}$, as you probably note.)

Two trivial examples: the least algebra $\{\emptyset, X\}$ and the greatest algebra 2^X.

1a11 Core exercise. If $\varphi : X \to Y$ and $\mathcal{F}$ is an algebra on Y then $\Phi(\mathcal{F})$ is an algebra on X. (As before, $\Phi = \varphi^{-1} : 2^Y \to 2^X$.)[2]

Prove it.

By (1a9) and 1a11,

$$(\mathcal{E} \cup \sim\!\mathcal{E})_{\mathrm{ds}} \text{ is an algebra on } X$$

whenever $\mathcal{E} \subset 2^X$ is finite.

1a12 Core exercise. The number of sets in a finite algebra is always of the form 2^k, $k = 0, 1, 2, \dots$, and every such 2^k corresponds to some finite algebra.[3] (Exclude $k = 0$ if you do not want X to be empty.)

Prove it.

1a13 Core exercise. The map $\varphi : X \to \{0,1\}^n$ given by (1a8) is injective (that is, one-to-one) if and only if $\mathcal{E}$ *separates points* (it means: whenever $x_1, x_2 \in X$ differ, there exists $A \in \mathcal{E}$ that contains exactly one of x_1, x_2).

Prove it.

1a14 Core exercise. If a finite algebra $\mathcal{E}$ separates points then $\mathcal{E} = 2^X$ (and X is necessarily finite).

Prove it.

Infinite $\mathcal{E}$ boils down to finite $\mathcal{E}$ as follows:

$$\text{(1a15)} \quad \mathcal{E}_{\mathrm{d}} = \bigcup_{\mathcal{F} \subset \mathcal{E}, \mathcal{F} \text{ is finite}} \mathcal{F}_{\mathrm{d}}\,, \quad \text{etc.,} \quad (\mathcal{E} \cup \sim\!\mathcal{E})_{\mathrm{ds}} = \bigcup_{\mathcal{F} \subset \mathcal{E}, \mathcal{F} \text{ is finite}} (\mathcal{F} \cup \sim\!\mathcal{F})_{\mathrm{ds}}\,.$$

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[1] Or "Boolean algebra of sets", or "concrete Boolean algebra", or "field of sets".

[2] Likewise, if $f : G_1 \to G_2$ is a homomorphism of groups and G is a subgroup of G_1 then $f(G)$ is a subgroup of G_2.

[3] Not only 2^{2^k}...

1a16 Core exercise. Prove that

$$(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{ds}} \text{ is an algebra on } X$$

whenever $\mathcal{E} \subset 2^X$ (not necessarily finite).

Thus, $(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{ds}}$ is the least algebra containing $\mathcal{E}$, in other words, the algebra *generated* by $\mathcal{E}$. A finite set generates a finite algebra; a countable set generates a countable algebra. We have the general form of a set from the generated algebra:

$$\bigcup_{i=1}^{I} \bigcap_{j=1}^{J_i} A_{i,j} \quad \text{for } A_{i,j} \in \mathcal{E} \cup {\sim}\mathcal{E} \,. \tag{1a17}$$

For a countably infinite $\mathcal{E} = \{A_1, A_2, \dots\}$ we may introduce $\varphi : X \to Y = \{0,1\}^\infty$ (infinite sequences) by

$$\varphi(x) = \big(\mathbf{1}_{A_1}(x), \mathbf{1}_{A_2}(x), \dots \big) \,, \tag{1a18}$$

and still,

$$(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{ds}} = \Phi\big((\mathcal{F} \cup {\sim}\mathcal{F})_{\mathrm{ds}} \big)$$

where $\mathcal{F} = \{B_1, B_2, \dots\} \subset 2^Y$, $B_k = \{ y \in Y : y(k) = 1 \}$ (as before, $\Phi = \varphi^{-1} : 2^Y \to 2^X$); but now $(\mathcal{F} \cup {\sim}\mathcal{F})_{\mathrm{ds}}$ is only a small part of 2^Y. Indeed, the former is countable, while the latter is not only uncountable but also exceeds the cardinality of continuum! Sets of $(\mathcal{F} \cup {\sim}\mathcal{F})_{\mathrm{ds}}$ are called *cylindrical* sets. They are exactly the sets "depending on finitely many coordinates each" (think, why).

In contrast to 1a14, the cylindrical algebra separates points but fails to contain all sets.

The set $\{0,1\}^\infty$ is basically the Cantor set $C \subset [0,1]$,

$$\{0,1\}^\infty \ni y \longleftrightarrow \sum_{k=1}^\infty \frac{2y(k)}{3^k} \in C \,. \tag{1a19}$$

1a20 Core exercise. The cylindrical algebra on the Cantor set is exactly the algebra of all clopen (that is, both closed and open) subsets.

Prove it.

For uncountable $\mathcal{E}$ we still have the cylindrical algebra on $\{0,1\}^{\mathcal{E}}$, but the latter is not the Cantor set.

Every algebra of sets is the inverse image of the algebra of cylindrical sets. In this sense the cylindrical algebra is universal.
In particular, (a) every finite algebra is the inverse image of the algebra of all subsets on a finite set; (b) every countable algebra is the inverse image of the algebra of all clopen subsets of the Cantor set.

Universal models are useful but not unavoidable. Do not use them when solving the next two exercises.

1a21 Core exercise. Prove that

$$\mathcal{E}_{\mathrm{ds}} = \mathcal{E}_{\mathrm{sd}}$$

whenever $\mathcal{E} \subset 2^X$.

1a22 Core exercise. Deduce 1a16 from 1a21.

1b Sigma-algebra

Sigma-algebra is no easy matter. Sigma-algebra generated by given sets is described only implicitly, but still, is tractable.

For arbitrary $\mathcal{E} \subset 2^X$ we denote

$$\begin{aligned}\mathcal{E}_\delta &= \{A_1 \cap A_2 \cap \dots : A_1, A_2, \dots \in \mathcal{E}\} \cup \{X\}\,,\\ \mathcal{E}_\sigma &= \{A_1 \cup A_2 \cup \dots : A_1, A_2, \dots \in \mathcal{E}\} \cup \{\emptyset\}\,.\end{aligned}$$

Clearly, $\mathcal{E}_\delta \supset \mathcal{E}$ and $\mathcal{E}_\sigma \supset \mathcal{E}$. Also, $\mathcal{E}_{\delta\delta} = \mathcal{E}_\delta$, $\mathcal{E}_{\sigma\sigma} = \mathcal{E}_\sigma$. And $({\sim}\mathcal{E})_\delta = {\sim}(\mathcal{E}_\sigma)$, $({\sim}\mathcal{E})_\sigma = {\sim}(\mathcal{E}_\delta)$. If $\mathcal{E}$ is finite then $\mathcal{E}_\delta = \mathcal{E}_{\mathrm{d}}$ and $\mathcal{E}_\sigma = \mathcal{E}_{\mathrm{s}}$ (still finite).

1b1 Core exercise. Prove that $\mathcal{E}_{\sigma\mathrm{d}} \subset \mathcal{E}_{\mathrm{d}\sigma}$ and $\mathcal{E}_{\delta\mathrm{s}} \subset \mathcal{E}_{\mathrm{s}\delta}$.

1b2 Extra exercise. Do the equalities $\mathcal{E}_{\sigma\mathrm{d}} = \mathcal{E}_{\mathrm{d}\sigma}$, $\mathcal{E}_{\delta\mathrm{s}} = \mathcal{E}_{\mathrm{s}\delta}$ hold in general, or not?

1b3 Example. $X = \{0,1\}^\infty$ (that is, the Cantor set) and $\mathcal{E}$ is the algebra of all cylindrical sets (that is, clopen sets, recall 1a20). Note that $\mathcal{E}$ is countable.

1b4 Core exercise. Let $X, \mathcal{E}$ be as in 1b3. Prove that for every $p \in [0,1]$ the set

$$A_p = \Big\{ x \in X : \frac{x(1) + \dots + x(n)}{n} \xrightarrow[n\to\infty]{} p \Big\}$$

belongs to $\mathcal{E}_{\delta\sigma\delta}$.

Generally, nothing useful can be said about an *uncountable* union of (say) $\mathcal{E}_{\delta\sigma\delta}$ sets. But nevertheless…

1b5 Extra exercise. Let $X, \mathcal{E}$ and A_p be as in 1b4. Prove that the set $A = \cup_{p \in [0,1]} A_p$ belongs to $\mathcal{E}_{\delta\sigma\delta}$.

1b6 Extra exercise. Let $X, \mathcal{E}$ be as in 1b3, and A the set of all $x \in X$ such that the series

$$\sum_{n=1}^{\infty} \frac{2x(n) - 1}{n}$$

converges. Prove that A belongs to $\mathcal{E}_{\delta\sigma\delta}$.

It is rather easy to prove that a given set belongs to the corresponding class. It is much harder to prove that it does *not* belong to another class.

1b7 Core exercise. Let $X, \mathcal{E}$ be as in 1b3. Prove that $\mathcal{E}_\delta$ is the set of all closed subsets of the Cantor set, and $\mathcal{E}_\sigma$ is the set of all open subsets of the Cantor set.

We see that countability of $\mathcal{E}$ does not imply countability of $\mathcal{E}_\delta, \mathcal{E}_\sigma$.

Sometimes one denotes (for $X, \mathcal{E}$ as in 1b3) $\mathcal{E}_\delta = F$ (closed sets) and $\mathcal{E}_\sigma = G$ (open sets); thus, $\mathcal{E}_{\delta\sigma} = F_\sigma$ (countable unions of closed sets) and $\mathcal{E}_{\sigma\delta} = G_\delta$ (countable intersections of open sets). The symbols F_σ, G_δ are widely used (not only in the context of the Cantor set).[1] Clearly, $\sim(F_\sigma) = G_\delta$ and $\sim(G_\delta) = F_\sigma$.

Do not think that (similarly to 1a21) $\mathcal{E}_{\delta\sigma} = \mathcal{E}_{\sigma\delta}$; it is not! If A is a *dense* F_σ set and its complement B is a *dense* G_δ set then A cannot be G_δ set, and B cannot be F_σ set (which follows easily from the famous Baire category theorem). In particular, a dense countable subset of the Cantor set is always F_σ and never G_δ.

1b8 Definition. A set $\mathcal{E} \subset 2^X$ is a *σ-algebra*[2] (on X) if $X \setminus A$, $A_1 \cap A_2 \cap \dots$ and $A_1 \cup A_2 \cup \dots$ belong to $\mathcal{E}$ for all $A, A_1, A_2, \dots \in \mathcal{E}$, and $\emptyset, X \in \mathcal{E}$. In other words, if

$$\sim\mathcal{E} \subset \mathcal{E}\,, \quad \mathcal{E}_\delta \subset \mathcal{E}\,, \quad \mathcal{E}_\sigma \subset \mathcal{E}\,.$$

(Clearly, the conditions $\mathcal{E}_\delta \subset \mathcal{E}$, $\mathcal{E}_\sigma \subset \mathcal{E}$ are equivalent, given $\sim\mathcal{E} \subset \mathcal{E}$.)

Two trivial examples: the least algebra $\{\emptyset, X\}$ and the greatest algebra 2^X are also the least σ-algebra and the greatest σ-algebra. Every *finite* algebra is a σ-algebra. Every σ-algebra is an algebra.

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[1] The symbols F, G are used more often for individual closed and open sets rather than the sets of all such sets.

[2] Probabilists often prefer "σ-field".

In contrast to 1a16, $(\mathcal{E} \cup {\sim}\mathcal{E})_{\delta\sigma}$ is generally not a σ-algebra (even for $X, \mathcal{E}$ of 1b3). In contrast to (1a17), the formula

$$\bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{i,j} \quad \text{for } A_{i,j} \in \mathcal{E} \cup {\sim}\mathcal{E} \, ,$$

the general form of a set from $(\mathcal{E} \cup {\sim}\mathcal{E})_{\delta\sigma}$, does not represent a σ-algebra.

As you probably know, a better situation appears when a measure is given and null sets are neglected; that is, equivalence classes are used rather than sets. In that framework, for an algebra $\mathcal{E}$, $\mathcal{E}_{\delta\sigma} = \mathcal{E}_{\sigma\delta}$ becomes a σ-algebra, — very convenient if you work in $\mathbb{R}^n$ with Lebesgue measure. However, in an infinite-dimensional space we typically have nothing like Lebesgue measure and, worse, no appropriate class of negligible sets. Rather, we have various measures that typically are singular to each other.

Back to our framework: what could we mean by a σ-algebra generated by a set $\mathcal{E}$ or, equally well, by an algebra $\mathcal{E}$? It appears that $\mathcal{E}_{\delta\sigma\delta\sigma}$ is still not a σ-algebra. In order to avoid clumsy notation like $\mathcal{E}_{\underbrace{\delta\sigma\dots\delta\sigma}_{100}}$ one may introduce $\Sigma_n = \Sigma_n(X, \mathcal{E}) \subset 2^X$ and $\Pi_n = \Pi_n(X, \mathcal{E}) \subset 2^X$ recursively:

(1b9) $$\Sigma_{n+1} = (\Pi_n)_\sigma \text{ and } \Pi_{n+1} = (\Sigma_n)_\delta \quad \text{for } n = 0, 1, 2, \dots$$

and $\Pi_0 = \mathcal{E}$, $\Sigma_0 = {\sim}\mathcal{E}$ for a given set $\mathcal{E} \subset 2^X$ satisfying

(1b10) $${\sim}\mathcal{E} \subset \mathcal{E}_\sigma$$

(which evidently holds when ${\sim}\mathcal{E} = \mathcal{E}$). Thus,

(1b11) $$\begin{gathered} \Sigma_1 = \mathcal{E}_\sigma, \quad \Sigma_2 = ({\sim}\mathcal{E})_{\delta\sigma}, \quad \Sigma_3 = \mathcal{E}_{\sigma\delta\sigma}, \dots \\ \Pi_1 = ({\sim}\mathcal{E})_\delta, \quad \Pi_2 = \mathcal{E}_{\sigma\delta}, \quad \Pi_3 = ({\sim}\mathcal{E})_{\delta\sigma\delta}, \dots \end{gathered}$$

1b12 Core exercise. Prove that ${\sim}\Sigma_n = \Pi_n$ for $n = 0, 1, 2, \dots$

1b13 Core exercise. Prove that $\Pi_n \cup \Sigma_n \subset \Pi_{n+1} \cap \Sigma_{n+1}$ for $n = 0, 1, 2, \dots$

1b14 Core exercise. Prove that $(\Pi_n)_{\text{ds}} = \Pi_n$ and $(\Sigma_n)_{\text{ds}} = \Sigma_n$ for $n = 2, 3, \dots$

(If $\mathcal{E}$ is an algebra, these equalities hold also for $n = 0, 1$, but generally they do not.)

1b15 Core exercise. Prove that $\Pi_n \cap \Sigma_n$ is an algebra for $n = 2, 3, \dots$

1b16 Core exercise. Prove that $\cup_n \Pi_n = \cup_n \Sigma_n = \cup_n (\Pi_n \cap \Sigma_n)$ is an algebra.

It appears that generally (and even for $X, \mathcal{E}$ of 1b3, see Sect. 1c) all these Π_n, Σ_n differ and none of them is a σ-algebra. Moreover, the algebra $\cup_n \Pi_n = \cup_n \Sigma_n$ is not a σ-algebra!

A better situation appears in algebra (recall generated subgroups, linear subspaces etc.) since an algebraic operation takes finitely many (usually, two) operands. The problem is that our operation $(A_1, A_2, \dots) \mapsto A_1 \cup A_2 \cup \dots$ takes infinitely many operands.

Fortunately, we have a completely different approach.

1b17 Definition. The σ-algebra $\sigma(\mathcal{E})$ *generated* by a set $\mathcal{E} \subset 2^X$ is the intersection of all σ-algebras that contain $\mathcal{E}$.

The intersection of σ-algebras (no matter how many) is a σ-algebra (think, why); at least one σ-algebra containing $\mathcal{E}$ exists (just 2^X); thus, the generated σ-algebra is well-defined. Clearly, $\sigma(\mathcal{E})$ is the least σ-algebra containing $\mathcal{E}$.

This definition is formally simple, but exploits the set theory quite heavily. In the huge set 2^{2^X} we choose the subset of all σ-algebras containing $\mathcal{E}$ (have you a clear idea of this subset?) and intersect them all!

Bad news: we have no useful general form of a set from the generated σ-algebra. It is usually not difficult to prove that a given set belongs to $\sigma(\mathcal{E})$ (when it does), since it usually appears to belong to Π_n or Σ_n for $n = 1, 2, 3$ (hardly 4). However, it is usually difficult to prove that a given set does not belong to $\sigma(\mathcal{E})$ (when it does not). Well, we try to percolate to useful results, avoiding hard obstacles. . .

1b18 Core exercise. For an uncountable $\mathcal{E}$,

$$\sigma(\mathcal{E}) = \bigcup_{\mathcal{F} \subset \mathcal{E}, \mathcal{F} \text{ is countable}} \sigma(\mathcal{F}) = \bigcup_{A_1, A_2, \dots \in \mathcal{E}} \sigma(A_1, A_2, \dots) .$$

Prove it.

In the next five exercises $\varphi : X \to Y$ and $\Phi = \varphi^{-1} : 2^Y \to 2^X$. Here are counterparts of 1a7 and 1a11.

1b19 Core exercise. Prove that

$$\Phi(\mathcal{F}_\delta) = (\Phi(\mathcal{F}))_\delta , \quad \Phi(\mathcal{F}_\sigma) = (\Phi(\mathcal{F}))_\sigma$$

for all $\mathcal{F} \subset 2^Y$.

1b20 Core exercise. If $\mathcal{F}$ is a σ-algebra on Y then $\Phi(\mathcal{F})$ is a σ-algebra on X.

Prove it.

1b21 Core exercise. If $\mathcal{E}$ is a σ-algebra on X then $\Phi^{-1}(\mathcal{E})$ is a σ-algebra on Y.[1]

Prove it.

1b22 Core exercise. Prove that $\Phi\bigl(\sigma(\mathcal{F})\bigr) \supset \sigma\bigl(\Phi(\mathcal{F})\bigr)$ for all $\mathcal{F} \subset 2^Y$.

1b23 Core exercise. Prove that $\Phi\bigl(\sigma(\mathcal{F})\bigr) \subset \sigma\bigl(\Phi(\mathcal{F})\bigr)$ for all $\mathcal{F} \subset 2^Y$.

Thus,

$$\Phi\bigl(\sigma(\mathcal{F})\bigr) = \sigma\bigl(\Phi(\mathcal{F})\bigr) \tag{1b24}$$

whenever $\varphi : X \to Y$ and $\mathcal{F} \subset 2^Y$. (As before, $\Phi = \varphi^{-1} : 2^Y \to 2^X$.)

1c Borel sets

Borel subsets of $\mathbb{R}^d$ are the most useful sigma-algebra. An infinite sequence of complexity levels fails to exhaust their hierarchy. The proof of this fact is explicit but not simple, it involves Cantor's diagonal argument and coding of sets by points of the Cantor set.

1c1 Definition. The *Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$* on $\mathbb{R}^d$ is the σ-algebra generated by open sets. Elements of $\mathcal{B}(\mathbb{R}^d)$ are called Borel subsets of $\mathbb{R}^d$.

Clearly, an arbitrary finite-dimensional linear space (over $\mathbb{R}$) can be used instead of $\mathbb{R}^d$.

1c2 Core exercise. For $X \subset \mathbb{R}^d$, the set $\{B \cap X : B \in \mathcal{B}(\mathbb{R}^d)\}$ is a σ-algebra on X generated by the set $\{G \cap X : G \text{ open in } \mathbb{R}^d\}$ of all relatively open sets in X.

Prove it.

The σ-algebra of 1c2 is called the Borel σ-algebra of X, and denoted $\mathcal{B}(X)$.

1c3 Core exercise. If X is a Borel set in $\mathbb{R}^d$ then $\mathcal{B}(X) = \{A \in \mathcal{B}(\mathbb{R}^d) : A \subset X\}$; otherwise it is not.

Prove it.

1c4 Core exercise. Prove that $\mathcal{B}(\mathbb{R})$ is generated by open intervals (a, b) for rational a, b. (That is, $\mathcal{B}(\mathbb{R})$ is *equal* to the σ-algebra generated by these intervals.)

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[1] Likewise, if $f : G_1 \to G_2$ is a homomorphism of groups and G is a subgroup of G_2 then $f^{-1}(G)$ is a subgroup of G_1.

1c5 Core exercise. Prove that $\mathcal{B}(\mathbb{R})$ is generated by closed intervals.

1c6 Core exercise. Prove that $\mathcal{B}(\mathbb{R}^2)$ is generated by open disks.

1c7 Core exercise. Prove that $\mathcal{B}(\mathbb{R}^2)$ is generated by vertical and horizontal open strips, $(a,b) \times \mathbb{R}$ and $\mathbb{R} \times (a,b)$ for all a, b, $a < b$.

1c8 Extra exercise. Let $A \subset \mathbb{R}^2$ be a bounded open neighborhood of the origin. Prove that $\mathcal{B}(\mathbb{R}^2)$ is generated by $\{x + rA : x \in \mathbb{R}^2, r \in (0, \infty)\}$ (here $x + rA = \{x + ra : a \in A\}$).

1c9 Extra exercise. Let $\mathbb{C}$ be the complex plane and $A \subset \mathbb{C}$ a set that has (at least one) interior point and (at least one) exterior point. Prove that $\mathcal{B}(\mathbb{C})$ is contained in the σ-algebra generated by $\{u + vA : u, v \in \mathbb{C}, v \ne 0\}$.

1c10 Core exercise. Prove that the Borel σ-algebra of the Cantor set (treated as a subset of $\mathbb{R}$) is generated by clopen sets.

Similarly to (1b9) we introduce $\Pi_n = \Pi_n(\mathbb{R}^d)$ and $\Sigma_n = \Sigma_n(\mathbb{R}^d)$ recursively:

$$\Sigma_{n+1} = (\Pi_n)_\sigma \text{ and } \Pi_{n+1} = (\Sigma_n)_\delta \quad \text{for } n = 1, 2, 3, \dots \tag{1c11}$$

This time, however, we start with Π_1, Σ_1 (rather than Π_0, Σ_0): Σ_1 is the set of all open sets, and Π_1 — closed sets. Similarly, for $X \subset \mathbb{R}^d$ we introduce $\Pi_n(X)$ and $\Sigma_n(X)$ using relatively open sets $(G \cap X)$ as Σ_1 and relatively closed sets $(F \cap X)$ as Π_1. Thus,

$$\begin{aligned} &\Sigma_1 = G, \quad \Sigma_2 = F_\sigma, \quad \Sigma_3 = G_{\delta\sigma}, \dots \\ &\Pi_1 = F, \quad \Pi_2 = G_\delta, \quad \Pi_3 = F_{\sigma\delta}, \dots \end{aligned} \tag{1c12}$$

We have

$$\Sigma_1 \subset (\Pi_1)_\sigma = \Sigma_2$$

(think, why); using it instead of (1b10) we get 1b12–1b14 as before, but for $n = 1, 2, \dots$:

$$\sim\Sigma_n = \Pi_n, \quad \Pi_n \cup \Sigma_n \subset \Pi_{n+1} \cap \Sigma_{n+1}, \quad (\Pi_n)_{\text{ds}} = \Pi_n, \quad (\Sigma_n)_{\text{ds}} = \Sigma_n. \tag{1c13}$$

1c14 Extra exercise. For *every* function $\mathbb{R} \to \mathbb{R}$, the set of its continuity points belongs to $\Pi_2(\mathbb{R})$.

Prove it.

1c15 Core exercise. If $X \subset \mathbb{R}^{d_1}$ and $\varphi : X \to \mathbb{R}^d$ is a continuous map then

$$\Phi\big(\Pi_n(\mathbb{R}^d)\big) \subset \Pi_n(X) \,, \quad \Phi\big(\Sigma_n(\mathbb{R}^d)\big) \subset \Sigma_n(X) \,;$$

as before, $\Phi = \varphi^{-1} : 2^{\mathbb{R}^d} \to 2^X$. If, in addition, φ is a homeomorphism (of X to $\varphi(X)$) then

$$\Phi\big(\Pi_n(\mathbb{R}^d)\big) = \Pi_n(X) \,, \quad \Phi\big(\Sigma_n(\mathbb{R}^d)\big) = \Sigma_n(X) \,.$$

Prove it.

1c16 Core exercise. If $X \subset \mathbb{R}^d$ then

$$\Pi_n(X) = \{ A \cap X : A \in \Pi_n(\mathbb{R}^d) \} \,, \quad \Sigma_n(X) = \{ A \cap X : A \in \Sigma_n(\mathbb{R}^d) \} \,.$$

Prove it.

1c17 Core exercise. If $X \subset \mathbb{R}^d$ and $A \subset X$ then

$$A \in \Pi_n(X) \;\wedge\; X \in \Pi_m(R^d) \quad \Longrightarrow \quad A \in \Pi_{\max(m,n)}(\mathbb{R}^d) \,,$$

and the same for $\Sigma(\dots)$.

Prove it.

Treating $X = \{0,1\}^\infty$ as (a copy of) the Cantor set C (recall (1a19)) we know that the cylindrical algebra $\mathcal{E}$ on $\{0,1\}^\infty$ is the clopen algebra on C (recall 1a20). Also, $\Sigma_1(X, \mathcal{E}) = \mathcal{E}_\sigma = \Sigma_1(C)$ is the set of all open sets (recall 1b7) whence (by induction)

$$\Pi_n(X,\mathcal{E}) = \Pi_n(C) \,, \quad \Sigma_n(X,\mathcal{E}) = \Sigma_n(C) \quad \text{for } n = 1,2,\dots$$

1c18 Theorem (Lebesgue 1905)**.**

$$\Pi_n(C) \ne \Sigma_n(C) \quad \text{for } n = 1,2,\dots$$

The theorem states that $F \ne G$ (evident), $G_\delta \ne F_\sigma$ (follows easily from the Baire category theorem), $F_{\sigma\delta} \ne G_{\delta\sigma}$ (did you know?), $G_{\delta\sigma\delta} \ne F_{\sigma\delta\sigma}$ (wow!), and so on.

Equivalently,

$$\Pi_n(X,\mathcal{E}) \ne \Sigma_n(X,\mathcal{E}) \quad \text{for } n = 1,2,\dots$$

where $X = \{0,1\}^\infty$ and $\mathcal{E}$ is the cylindrical algebra. That is, $\mathcal{E}_\delta \ne \mathcal{E}_\sigma$, $\mathcal{E}_{\sigma\delta} \ne \mathcal{E}_{\delta\sigma}$, $\mathcal{E}_{\delta\sigma\delta} \ne \mathcal{E}_{\sigma\delta\sigma}$ and so on.

The proof is a wonderful reincarnation of the famous Cantor's diagonal argument.[1] Let us recall this argument.

THEOREM. It is impossible to map a set X *onto* the set 2^X.

PROOF. Let $f : X \to 2^X$. We define $A \subset X$ by

$$\forall x \; \big(x \in A \quad \Longleftrightarrow \quad x \notin f(x) \big) \, .$$

It cannot happen that $A = f(x_0)$ for some $x_0 \in X$, since this would imply

$$x \in f(x_0) \quad \Longleftrightarrow \quad x \in A \quad \Longleftrightarrow \quad x \notin f(x)$$

for all x, in particular, for $x = x_0$,

$$x_0 \in f(x_0) \quad \Longleftrightarrow \quad x_0 \notin f(x_0) \, ;$$

a contradiction.

Treating x as a code of the set $f(x)$ we interpret the crucial relation $x \notin f(x)$ as

"the set encoded by x does not contain x".

Keeping this phrase in mind, we'll encode sets of Σ_n by points of the Cantor set.

For now X is arbitrary, and $\mathcal{E} \subset 2^X$ is countable, otherwise arbitrary. We enumerate it: $\mathcal{E} = \{E_1, E_2, \dots\}$.

The general form of a set $A \in \mathcal{E}_\sigma$ is, of course, $A = A_1 \cup A_2 \cup \dots$ where $A_1, A_2, \dots \in \mathcal{E}$. However, we need another general form. We define $\xi_1 : \{0,1\}^\infty \to \mathcal{E}_\sigma$ as follows: for all $x \in X$,

$$x \in \xi_1(t) \quad \Longleftrightarrow \quad \exists n \, \big(x \in E_n \wedge t(n) = 1 \big) \, .$$

1c19 Core exercise. Prove that ξ_1 maps $\{0,1\}^\infty$ *onto* $\mathcal{E}_\sigma$.

Further we introduce the set $\{0,1\}^{\infty\times\infty} = \{0,1\}^{\infty^2}$ of all two-dimensional arrays t of numbers $t(m,n) \in \{0,1\}$ given for $m,n \in \{1,2,\dots\}$. (The notation $\infty \times \infty$ instead of $\{1,2,\dots\} \times \{1,2,\dots\}$ is informal but convenient). We define $\xi_2 : \{0,1\}^{\infty\times\infty} \to \mathcal{E}_{\sigma\delta}$ as follows: for all $x \in X$,

$$x \in \xi_2(t) \quad \Longleftrightarrow \quad \forall m \, \exists n \, \big(x \in E_n \wedge t(m,n) = 1 \big) \, .$$

1c20 Core exercise. Prove that ξ_2 maps $\{0,1\}^{\infty\times\infty}$ *onto* $\mathcal{E}_{\sigma\delta}$.

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[1] More reincarnations: Gödel's first incompleteness theorem; undecidability of the halting problem.

In the same way, $\xi_3 : \{0,1\}^{\infty^3} \to \mathcal{E}_{\sigma\delta\sigma}$,

$$x \in \xi_3(t) \quad \Longleftrightarrow \quad \exists l \, \forall m \, \exists n \, \big(x \in E_n \wedge t(l,m,n) = 1 \big) \, ,$$

and so on.

We need a code in $\{0,1\}^\infty$ rather than $\{0,1\}^{\infty^3}$. But this is not a problem: anyway it is just $\{0,1\}^{(\text{a countable set})}$. We choose bijections $f_2 : \{1,2,\dots\} \times \{1,2,\dots\} \to \{1,2,\dots\}$, $f_3 : \{1,2,\dots\}^3 \to \{1,2,\dots\}$ and so on. We treat $t \in \{0,1\}^\infty$ as the code of the set $\xi_1(t) \in \mathcal{E}_\sigma$, but also of the set $\xi_2(t \circ f_2) \in \mathcal{E}_{\sigma\delta}$, and $\xi_3(t \circ f_3) \in \mathcal{E}_{\sigma\delta\sigma}$ and so on. All sets of these classes have codes. We note that

$$\begin{aligned} x \in \xi_1(t) \quad &\Longleftrightarrow \quad \exists n \, \big(x \in E_n \wedge t(n) = 1 \big) \, , \\ x \in \xi_2(t \circ f_2) \quad &\Longleftrightarrow \quad \forall m \, \exists n \, \big(x \in E_n \wedge t(f_2(m,n)) = 1 \big) \, , \\ x \in \xi_3(t \circ f_3) \quad &\Longleftrightarrow \quad \exists l \, \forall m \, \exists n \, \big(x \in E_n \wedge t(f_3(l,m,n)) = 1 \big) \end{aligned} \tag{1c21}$$

and so on. The formulas above implement the phrase "the set encoded by t contains x".

Now we return to $X, \mathcal{E}$ of (the equivalent formulation of) Theorem 1c18: $X = \{0,1\}^\infty$ and $\mathcal{E}$ is the algebra of all cylindrical sets. The phrase "the set encoded by x does not contain x" is implemented as follows:

$$\begin{aligned} &\neg \exists n \, \big(x \in E_n \wedge x(n) = 1 \big) \, , && \text{for } \mathcal{E}_\sigma \\ &\neg \forall m \, \exists n \, \big(x \in E_n \wedge x(f_2(m,n)) = 1 \big) \, , && \text{for } \mathcal{E}_{\sigma\delta} \\ &\neg \exists l \, \forall m \, \exists n \, \big(x \in E_n \wedge x(f_3(l,m,n)) = 1 \big) && \text{for } \mathcal{E}_{\sigma\delta\sigma} \end{aligned}$$

and so on. (Here "$\neg$" is the negation.)

1c22 Core exercise. Prove that the set $A_1 \subset X$ defined by

$$\forall x \, \big(\, x \in A_1 \quad \Longleftrightarrow \quad \neg \exists n \, (x \in E_n \wedge x(n) = 1) \, \big)$$

belongs to $\sim(\mathcal{E}_\sigma)$.

1c23 Core exercise. Prove that the set $A_2 \subset X$ defined by

$$\forall x \, \big(\, x \in A_2 \quad \Longleftrightarrow \quad \neg \forall m \, \exists n \, (x \in E_n \wedge x(f_2(m,n)) = 1) \, \big)$$

belongs to $\sim(\mathcal{E}_{\sigma\delta})$.

In the same way, the set A_3 defined by

$$\forall x \, \big(\, x \in A_3 \quad \Longleftrightarrow \quad \neg \exists l \, \forall m \, \exists n \, (x \in E_n \wedge x(f_3(l,m,n)) = 1) \, \big)$$

belongs to $\sim(\mathcal{E}_{\sigma\delta\sigma})$; and so on.

Finally, $A_1 \notin \mathcal{E}_\sigma$, since otherwise $A_1 = \xi_1(t)$ for some t (all sets have codes!), and therefore by (1c21), for all x

$$x \in A_1 \iff x \in \xi_1(t) \iff \exists n \bigl(x \in E_n \wedge t(n) = 1 \bigr) \,,$$

which contradicts the definition of A_1 when $x = t$.

Similarly, $A_2 \notin \mathcal{E}_{\sigma\delta}$, since otherwise $A_2 = \xi_2(t \circ f_2)$ for some t, and therefore by (1c21), for all x

$$x \in A_2 \iff x \in \xi_2(t \circ f_2) \iff \forall m \, \exists n \bigl(x \in E_n \wedge t(f_2(m,n)) = 1 \bigr) \,,$$

which contradicts the definition of A_2 when $x = t$.

In the same way $A_3 \notin \mathcal{E}_{\sigma\delta\sigma}$, and so on.

Theorem 1c18 is thus proved.[1]

Now we are in position to prove that

$$\Pi_n \cup \Sigma_n \subsetneqq \Pi_{n+1} \cap \Sigma_{n+1} \,. \tag{1c24}$$

Denoting the left half of the Cantor set C by C_0 and the right half by C_1 we observe that C_0, C_1 are homeomorphic to $C = C_0 \uplus C_1$. (In terms of $\{0,1\}^\infty$ it means $X_0 = \{x : x(1) = 0\}$ and $X_1 = \{x : x(1) = 1\}$.) Thus (recall 1c15) $\Pi_n(C_0) \ne \Sigma_n(C_0)$, $\Pi_n(C_1) \ne \Sigma_n(C_1)$. We take $A_0 \in \Pi_n(C_0) \setminus \Sigma_n(C_0)$, $A_1 \in \Sigma_n(C_1) \setminus \Pi_n(C_1)$ and $A = A_0 \cup A_1$. We note that A_0, A_1 belong to the algebra $\Pi_{n+1}(C) \cap \Sigma_{n+1}(C)$ (recall 1b15). However, $A \notin \Pi_n(C) \cup \Sigma_n(C)$, which proves (1c24).

The same set A may be treated as a subset of $\mathbb{R}^d$ (since $C \subset \mathbb{R} \subset \mathbb{R}^d$).

1c25 Core exercise. Prove that $A \in \Pi_{n+1}(\mathbb{R}^d) \cap \Sigma_{n+1}(\mathbb{R}^d)$.

1c26 Core exercise. Prove that $A \notin \Pi_n(\mathbb{R}^d) \cup \Sigma_n(\mathbb{R}^d)$.

We see that

$$\Pi_n(\mathbb{R}^d) \cup \Sigma_n(\mathbb{R}^d) \subsetneqq \Pi_{n+1}(\mathbb{R}^d) \cap \Sigma_{n+1}(\mathbb{R}^d) \quad \text{for } n = 1, 2, \dots \tag{1c27}$$

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[1] You may say: no, rather, for every n separately the claim "$\Pi_n(X,\mathcal{E}) \ne \Sigma_n(X,\mathcal{E})$" is proved (and the quantifier complexity of the proof depends on n). We still do not have a proof of the claim "$\forall n \, \Pi_n(X,\mathcal{E}) \ne \Sigma_n(X,\mathcal{E})$".

If you understand the problem, you should be able to solve it. To this end, define (by a single definition) the sequence $(f_n)_n$ of maps $f_n : X^{\{1,2,\dots\}^{2n}} \to X$ such that

$$f_n(x) = \cap_{i_1} \cup_{j_1} \cdots \cap_{i_n} \cup_{j_n} x_{i_1,j_1,\dots,i_n,j_n} \quad \text{for } x = \bigl(x_{i_1,j_1,\dots,i_n,j_n} \bigr)_{i_1,j_1,\dots,i_n,j_n} \,.$$

and therefore

$$\Pi_n(\mathbb{R}^d) \subsetneqq \Pi_{n+1}(\mathbb{R}^d)\,, \quad \Sigma_n(\mathbb{R}^d) \subsetneqq \Sigma_{n+1}(\mathbb{R}^d)\,. \tag{1c28}$$

The same holds for every closed $X \subset \mathbb{R}^d$ that contains a homeomorphic copy of the Cantor set. (In fact, every uncountable closed set does.)

Finally we prove that (recall 1b16)

$$\text{the algebra } \cup_n \Sigma_n(C) \text{ is } \textit{not} \text{ a } \sigma\text{-algebra.} \tag{1c29}$$

To this end we choose infinitely many disjoint clopen subsets $C_1, C_2, \dots \subset C$ homeomorphic to C (in terms of $X = \{0,1\}^\infty$ we may take $X_k = \{x : x(1) = \dots = x(k-1) = 0, x(k) = 1\}$). Then we choose $A_n \in \Sigma_n(C_n) \setminus \Sigma_{n-1}(C_n)$ and $A = A_1 \cup A_2 \cup \dots$ Clearly, $A \in \big(\cup_n \Sigma_n(C)\big)_\sigma$. However, $A \notin \cup_n \Sigma_n(C)$, since $A \in \Sigma_n(C)$ (for some n) would imply $A_{n+1} = A \cap C_{n+1} \in \Sigma_n(C_{n+1})$.

1c30 Core exercise. Prove that the algebra $\cup_n \Pi_n(\mathbb{R}^d) = \cup_n \Sigma_n(\mathbb{R}^d) = \cup_n \big(\Pi_n(\mathbb{R}^d) \cap \Sigma_n(\mathbb{R}^d)\big)$ is *not* a σ-algebra.

These Π_n, Σ_n are the so-called finite Borel hierarchy. Theorem 1c18 and its implications ("the hierarchy theorem") state that the finite Borel hierarchy does not collapse.[1]

1d Measurable spaces, measurable maps

Hopefully you are acquainted with some kinds of spaces (such as Euclidean spaces, Hilbert spaces, topological and metric spaces, measure spaces), but measurable spaces will probably surprise you.

1d1 Definition. A *measurable space* is a pair $(X, \mathcal{A})$ consisting of a set X and a σ-algebra $\mathcal{A}$ on X. Sets belonging to $\mathcal{A}$ are called *measurable.*

WARNING. In contrast to measure spaces, in this context (a) no measure is given; (b) no subset is called negligible (null); (c) measurability of a subset $A \subset X$ means just $A \in \mathcal{A}$.[2]

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[1] This hierarchy can be extended to the (transfinite) Borel hierarchy, indexed by all countable ordinals, but this is beyond our course. In fact, the hierarchy does not collapse on a countable ordinal (Lebesgue 1905). The whole Borel σ-algebra is reached only at the first uncountable ordinal.

[2] The phrase "measurable space" is sometimes avoided "as in fact many of the most interesting examples of such objects have no useful measures associated with them" (D.H. Fremlin, "Measure theory", Vol. 1, Sect. 111B).

1d2 Example. (a) $\mathbb{R}^d$ with its Borel σ-algebra; (b) the Cantor set with its Borel σ-algebra; (c) the set $\{0,1\}^\infty$ with the σ-algebra generated by cylindrical sets.

Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be measurable spaces.

1d3 Definition. A map $\varphi : X \to Y$ is called a *measurable* map from $(X,\mathcal{A})$ to $(Y,\mathcal{B})$, or just *measurable*, if $\varphi^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.

1d4 Core exercise. The composition of measurable maps is measurable. That is, if φ is a measurable map from $(X,\mathcal{A})$ to $(Y,\mathcal{B})$ and ψ is a measurable map from $(Y,\mathcal{B})$ to $(Z,\mathcal{C})$ then $x \mapsto \psi(\varphi(x))$ is a measurable map from $(X,\mathcal{A})$ to $(Z,\mathcal{C})$.

Prove it.

1d5 Definition. Measurable spaces $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ are called *isomorphic* if there exists a bijection $\varphi : X \to Y$ such that φ and φ^{-1} are measurable (such φ is called an *isomorphism*).

1d6 Core exercise. Prove that "isomorphic" is an equivalence relation between measurable spaces.

1d7 Example. Measurable spaces of 1d2(b,c) are evidently isomorphic. In fact, they are also isomorphic to 1d2(a) (irrespective of the dimension d), but this is far not evident.

1d8 Core exercise. Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be measurable spaces, and $\mathcal{B} = \sigma(\mathcal{F})$ (for a given $\mathcal{F} \subset 2^Y$). Prove that a map $\varphi : X \to Y$ is measurable if and only if $\varphi^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{F}$.

Whenever the σ-algebra $\mathcal{B}$ on Y is called the Borel σ-algebra, measurable maps $X \to Y$ are called Borel maps (or Borel measurable maps), as well as Borel functions (mostly for $Y = \mathbb{R}$).

Whenever $X \subset \mathbb{R}^d$, by default X is endowed with its Borel σ-algebra.

Whenever X is at most countable, by default X is endowed with the σ-algebra 2^X.

1d9 Core exercise. If $X \subset \mathbb{R}^d$ is at most countable then its Borel σ-algebra is equal to 2^X.

Prove it.

1d10 Core exercise. Let $(X,\mathcal{A})$ be a measurable space. Prove that a function $f : X \to \mathbb{R}$ is Borel if and only if $\{x : f(x) \le b\} \in \mathcal{A}$ for all $b \in \mathbb{R}$.

1d11 Core exercise. If $X \subset \mathbb{R}^{d_1}$ then every continuous map $X \to \mathbb{R}^d$ is Borel.

Prove it.

1d12 Core exercise. If $\varphi : X \to (0, \infty)$ is a Borel function then also $\frac{1}{\varphi} : x \mapsto \frac{1}{\varphi(x)}$ is a Borel function, and $x \mapsto \big(\varphi(x), \frac{1}{\varphi(x)}\big)$ is a Borel map $X \to \mathbb{R}^2$.

Prove it.

1d13 Definition. Let X be a set, $(Y, \mathcal{B})$ a measurable space, and $\varphi : X \to Y$. Then:

(a) The σ-algebra *generated* by φ is $\sigma(\varphi) = \Phi(\mathcal{B}) = \{\varphi^{-1}(B) : B \in \mathcal{B}\}$. (Recall 1b20; as before, $\Phi = \varphi^{-1} : 2^Y \to 2^X$.)

(b) The σ-algebra *generated* by s sequence of maps $\varphi_i : X \to Y$ is $\sigma(\varphi_1, \varphi_2, \dots) = \sigma\big(\sigma(\varphi_1) \cup \sigma(\varphi_2) \cup \dots\big) = \sigma\big(\Phi_1(\mathcal{B}) \cup \Phi_2(\mathcal{B}) \cup \dots\big) = \sigma\big(\{\varphi_i^{-1}(B) : B \in \mathcal{B}, i = 1, 2, \dots\}\big)$.

Likewise, $\sigma(\varphi_1, \varphi_2, \dots)$ is defined when $\varphi_i : X \to Y_i$, Y_i being endowed with $\mathcal{B}_i$. Also, i may run over an arbitrary index set (finite, countable, uncountable).

Similarly to 1b18, for an uncountable I,

$$\sigma\big(\{\varphi_i : i \in I\}\big) = \bigcup_{i_1, i_2, \dots \in I} \sigma(\varphi_{i_1}, \varphi_{i_2}, \dots) \,. \tag{1d14}$$

1d15 Definition. The *product* of two measurable spaces is a measurable space

$$(X, \mathcal{A}) \times (Y, \mathcal{B}) = (X \times Y, \mathcal{A} \times \mathcal{B}) \,,$$

where $\mathcal{A} \times \mathcal{B}$ is the σ-algebra generated by the two projection maps, $(x, y) \mapsto x$ and $(x, y) \mapsto y$.[1]

That is, $\mathcal{A} \times \mathcal{B} = \sigma\big(\{A \times Y : A \in \mathcal{A}\} \cup \{X \times B : B \in \mathcal{B}\}\big) = \sigma\big(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}\big)$. By default, $X \times Y$ is endowed by $\mathcal{A} \times \mathcal{B}$.

Likewise, the product of arbitrarily many measurable spaces $(X_i, \mathcal{A}_i)$ consists of the set $\tilde{X} = \prod_i X_i$ and the σ-algebra $\tilde{\mathcal{A}}$ generated by all projection maps $p_i : X \to X_i$, $p_i(x) = x(i)$.

In particular, taking $(X_i, \mathcal{A}_i) = (X, \mathcal{A})$ for all i we get the power, $(\tilde{X}, \tilde{\mathcal{A}}) = (X, \mathcal{A})^I$.

1d16 Core exercise. The measurable space $\{0, 1\}^\infty$ of 1d2(c) is the same as the product space $\{0, 1\} \times \{0, 1\} \times \dots$

Prove it.

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[1] It is often denoted by $\mathcal{A} \otimes \mathcal{B}$ rather than $\mathcal{A} \times \mathcal{B}$.

1d17 Core exercise. Prove that $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \times (\mathbb{R}, \mathcal{B}(\mathbb{R})) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

Similarly, $(\mathbb{R}^{d_1}, \mathcal{B}(\mathbb{R}^{d_1})) \times (\mathbb{R}^{d_2}, \mathcal{B}(\mathbb{R}^{d_2})) = (\mathbb{R}^{d_1+d_2}, \mathcal{B}(\mathbb{R}^{d_1+d_2}))$.

1d18 Core exercise. Let $(X, \mathcal{A})$, $(Y_1, \mathcal{B}_1)$ and $(Y_2, \mathcal{B}_2)$ be measurable spaces. Prove that a map $\varphi : X \to Y_1 \times Y_2$, $\varphi(x) = \big(\varphi_1(x), \varphi_2(x)\big)$, is measurable if and only if φ_1, φ_2 are measurable.

The same holds for $\prod_i (Y_i, \mathcal{B}_i)$.
Now reconsider 1d12...

1d19 Core exercise. If $\varphi, \psi : X \to \mathbb{R}^d$ are Borel maps then $\varphi + \psi$ is a Borel map. (Here $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$.)
Prove it.

1d20 Definition. A measurable space $(X, \mathcal{A})$ is *separated,* if $\mathcal{A}$ separates points, that is,

$$\forall x_1, x_2 \in X \, \big(\, x_1 \ne x_2 \quad \Longrightarrow \quad \exists A \in \mathcal{A} \, (x_1 \in A \wedge x_2 \notin A) \, \big) \, .$$

Equivalently,

$$\forall x_1, x_2 \in X \, \big(\forall A \in \mathcal{A} \, (x_1 \in A \iff x_2 \in A) \quad \Longrightarrow \quad x_1 = x_2 \, \big) \, .$$

(See also 1a13.)

1d21 Core exercise. If $(X_i, \mathcal{A}_i)$ is separated for every $i \in I$ then $\prod_{i \in I} (X_i, \mathcal{A}_i)$ is separated.
Prove it.

1d22 Core exercise. If X is at most countable and $(X, \mathcal{A})$ is separated then $\mathcal{A} = 2^X$.
Prove it.

Now reconsider 1d9...

1d23 Definition. A measurable space $(X, \mathcal{A})$ is *countably separated,* if $\mathcal{A}$ contains some at most countable set that separates points.

That is,

$$\forall x_1, x_2 \in X \, \big(\forall n \, (x_1 \in A_n \iff x_2 \in A_n) \quad \Longrightarrow \quad x_1 = x_2 \, \big)$$

for some $A_1, A_2, \dots \in \mathcal{A}$.

1d24 Core exercise. If I is at most countable and each $(X_i, \mathcal{A}_i)$ is countably separated then $\prod_{i \in I} (X_i, \mathcal{A}_i)$ is countably separated.

Prove it.

For uncountable I the product σ-algebra is quite weak. By 1d14, it contains only sets "depending on countably many coordinates each". More formally, let $(\tilde X, \tilde{\mathcal{A}}) = \prod_{i \in I} (X_i, \mathcal{A}_i)$, then

$$\tilde{\mathcal{A}} = \bigcup_{i_1, i_2, \dots \in I} \sigma(p_{i_1}, p_{i_2}, \dots) ;$$

as before, $p_i : \tilde X \to X_i$ are the projection maps.

1d25 Extra exercise. If I is uncountable and X contains more than one point then $(X, \mathcal{A})^I$ is *not* countably separated.

Prove it.

Thus, "separated" does not imply "countably separated".

1d26 Extra exercise. In the measurable space $[0,1]^{[0,1]}$ each of the following sets is *not* measurable:

- all Borel functions $[0,1] \to [0,1]$;
- all continuous functions $[0,1] \to [0,1]$;
- all increasing functions $[0,1] \to [0,1]$;
- all constant functions $[0,1] \to [0,1]$;
- the zero function $[0,1] \to [0,1]$ only.

Prove it.

A larger σ-algebra on $[0,1]^I$, the so-called Borel σ-algebra, is generated by all open sets (in the product topology). An open set in $[0,1]^I$ is the union of open cylindrical sets of the form

$$\{ x : x(i_1) \in (a_1, b_1), \dots, x(i_n) \in (a_n, b_n) \} ;$$

it is generally not a *countable* union, and so, an open set need not belong to the product σ-algebra.

1d27 Extra exercise. Each of the following sets belongs to the Borel σ-algebra on $[0,1]^{[0,1]}$:

- the zero function $[0,1] \to [0,1]$ only;
- all constant functions $[0,1] \to [0,1]$;
- all increasing functions $[0,1] \to [0,1]$;

$*$ all continuous functions $[0,1] \to [0,1]$.

Prove it.

About all Borel functions $[0,1] \to [0,1]$, I do not know. (I guess, it does not belong.)

1d28 Definition. A measurable space $(X, \mathcal{A})$ (as well as its σ-algebra $\mathcal{A}$) is *countably generated,* if $\mathcal{A} = \sigma(A_1, A_2, \dots)$ for some $A_1, A_2, \dots \in \mathcal{A}$.

Finitely generated σ-algebras are finite (think, why), but countably generated σ-algebras are generally uncountable.[1]

1d29 Core exercise. Prove that every subset of $\mathbb{R}^d$ (with its Borel σ-algebra) is a countably generated measurable space.

1d30 Core exercise. If I is at most countable and each $(X_i, \mathcal{A}_i)$ is countably generated then $\prod_{i\in I}(X_i, \mathcal{A}_i)$ is countably generated.

Prove it.

1d31 Extra exercise. If I is uncountable and $\mathcal{A} \ne \{\emptyset, X\}$ then $(X, \mathcal{A})^I$ is *not* countably generated.

Prove it.

1d32 Core exercise. If the σ-algebra $\mathcal{A} = \sigma(A_1, A_2, \dots)$ separates points then the sequence $A_1, A_2, \dots$ separates points.

Prove it.

1d33 Definition. A *Borel space* is a separated, countably generated measurable space.[2]

1d34 Core exercise. Every Borel space is countably separated.

Prove it.

1d35 Core exercise. $(X, \mathcal{A})$ is countably separated if and only if for some sub-σ-algebra $\mathcal{A}_1 \subset \mathcal{A}$ the measurable space $(X, \mathcal{A}_1)$ is a Borel space.

Prove it.

Recall the idea of (1a18): $\varphi : X \to \{0,1\}^\infty$,

$$\varphi(x) = \big(\mathbf{1}_{A_1}(x), \mathbf{1}_{A_2}(x), \dots\big) . \tag{1d36}$$

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[1] In fact, of cardinality not higher than continuum.

[2] Some authors define a Borel space as just a measurable space, not necessarily separated and countably generated.

1d37 Core exercise. A measurable space is countably separated if and only if it admits a measurable injection (that is, one-to-one map) into $\{0,1\}^\infty$ (or the Cantor set).

Prove it.

1d38 Core exercise. A measurable space $(X,\mathcal{A})$ is countably generated if and only if $\mathcal{A}$ is generated by some map $X \to \{0,1\}^\infty$.

Prove it.

1d39 Core exercise. A measurable space $(X,\mathcal{A})$ is a Borel space if and only if $\mathcal{A}$ is generated by some injection into $\{0,1\}^\infty$.

Prove it.

1d40 Core exercise. A measurable space is a Borel space if and only if it is isomorphic to a subset of $\mathbb{R}$ (with its Borel σ-algebra).

Prove it.

Clearly, $\mathbb{R}$ may be replaced with any $\mathbb{R}^d$, as well as with the Cantor set. (Thus, $\mathbb{R}^d$ is isomorphic to a subset of the Cantor set; compare it with 1d7).

It is trivial that every σ-algebra is the union of its countably generated sub-σ-algebras (since it evidently is the union of its at most four-element sub-σ-algebras). However, the following fact is worth to note.

1d41 Core exercise. Let $(X,\mathcal{A}) \times (Y,\mathcal{B}) = (Z,\mathcal{C})$, then $\mathcal{C}$ is the union of $\mathcal{A}_1 \times \mathcal{B}_1$ where $\mathcal{A}_1$ runs over all countably generated sub-σ-algebras of $\mathcal{A}$, and $\mathcal{B}_1$ — of $\mathcal{B}$.

Prove it.

1d42 Core exercise. Let $(X,\mathcal{A}) \times (Y,\mathcal{B}) = (Z,\mathcal{C})$, then every $C \in \mathcal{C}$ is of the form

$$C = (\varphi \times \psi)^{-1}(E)$$

for some measurable maps $\varphi : A \to \{0,1\}^\infty$, $\psi : B \to \{0,1\}^\infty$ and some measurable $E \subset \{0,1\}^\infty \times \{0,1\}^\infty$. Here $\varphi \times \psi : X \times Y \to \{0,1\}^\infty \times \{0,1\}^\infty$, $(\varphi \times \psi)(x,y) = \big(\varphi(x),\psi(y)\big)$.

Prove it.

Clearly, $\{0,1\}^\infty$ may be replaced with the Cantor set, or $\mathbb{R}$, or any $\mathbb{R}^d$.

Borel subsets of $\mathbb{R}^2$ (or of the square of the Cantor set) are a universal model for measurable sets in the product of two arbitrary measurable spaces.

Hints to exercises

1a3: first, try $n = 2, 3$.

1a6: $(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{d}}$ contains all singletons (single-point sets).

1a7: $\Phi(B_1 \cap B_2) = \Phi(B_1) \cap \Phi(B_2)$.

1a11: use 1a7.

1a12: k is the number of points in $\varphi(X) \subset \{0,1\}^n$.

1a14: use (1a9) and 1a13.

1a16: $\forall A, B \in (\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{ds}}\ \exists \mathcal{F}[\subset \mathcal{E},\ \text{finite}]\big(A, B \in (\mathcal{F} \cup {\sim}\mathcal{F})_{\mathrm{ds}}\big)$.

1a20: given a clopen $A \subset C$, take n such that the distance between A and $C \setminus A$ exceeds 3^{-n}.

1a21: open the brackets in $(A_1 \cup \dots \cup A_k) \cap (B_1 \cup \dots \cup B_l)$.

1a22: $(\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{dsd}} = (\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{sdd}} = (\mathcal{E} \cup {\sim}\mathcal{E})_{\mathrm{sd}}$.

1b1: open the brackets in $(A_1 \cup A_2 \cup \dots) \cap (B_1 \cup B_2 \cup \dots)$.

1b4: $x \in A_p$ if and only if

$$\forall \varepsilon > 0\ \exists n\ \forall m \quad \Big| \frac{x(1) + \dots + x(n+m)}{n+m} - p \Big| < \varepsilon\,.$$

1b7: $\mathcal{E}$ is a (countable) base of the topology on the Cantor set.

1b12: by induction.

1b13: by induction, using (1b10) and 1b12.

1b14: $(\Sigma_{n-1})_{\delta\mathrm{s}} \subset (\Sigma_{n-1})_{\mathrm{s}\delta}$ by 1b1.

1b15: use 1b14 and 1b12.

1b16: use 1b15.

1b18: similar to 1a15.

1b22: $\Phi(\sigma(\mathcal{F}))$ is a σ-algebra containing $\Phi(\mathcal{F})$.

1b23: denote $\mathcal{E} = \sigma(\Phi(\mathcal{F}))$, then $\mathcal{F} \subset \Phi^{-1}(\mathcal{E})$; $\sigma(\mathcal{F}) \subset \Phi^{-1}(\mathcal{E})$; $\Phi(\sigma(\mathcal{F})) \subset \mathcal{E}$.

1c2: apply (1b24) to the embedding $X \to \mathbb{R}^d$, $x \mapsto x$.

1c4: $\mathcal{E} \subset G \subset \mathcal{E}_\sigma$.

1c5: ${\sim}\mathcal{E} \subset G \subset \mathcal{E}_\sigma$.

1c6: $\mathcal{E} \subset G \subset \mathcal{E}_\sigma$.

1c7: $\mathcal{E} \subset G \subset \mathcal{E}_{\mathrm{d}\sigma}$.

1c10: use 1b7.

1c15: by induction, using 1b19 (and 1a7).

1c16: use 1c15.

1c17: use 1c16.

1c20: consider $A_m = \{x : \exists n(x \in E_n \wedge t(m,n) = 1)\} \in \mathcal{E}_\sigma$.

1c22: $\{x : x \in E_n \wedge x(n) = 1\} \in \mathcal{E}$ for all n.

1c23: $\{x : x \in E_n \wedge x(f_2(m,n)) = 1\} \in \mathcal{E}$ for all m, n.

1c25: use 1c17.

1c26: use 1c16.

1c30: recall the proof of (1c27).

1d6: use 1d4.

1d8: use 1b21.

1d9: a singleton (that is, single-point set) is closed.

1d10: use 1d8; recall 1c4, 1c5.

1d11: apply 1d8 to open sets.

1d12: use 1d4 and 1d11.

1d16: $\sigma(p_k) = \sigma(A_k)$, A_k as in 1a2.

1d17: use 1c7.

1d18: use 1d8.

1d19: the map $(x,y) \mapsto x + y$ is continuous, therefore Borel.

1d21: $\forall i \big(p_i(x_1) = p_i(x_2) \big) \quad \Longrightarrow \quad x_1 = x_2$.

1d22: each singleton is the intersection of some sequence of measurable sets.

1d24: $\mathcal{E} = \cup_i p_i^{-1}(\mathcal{E}_i)$ is countable.

1d29: recall 1c4.

1d30: $\mathcal{E} = \cup_i p_i^{-1}(\mathcal{E}_i)$ is countable.

1d32: $\{A : x_1 \in A \iff x_2 \in A\}$ is a σ-algebra (for given x_1, x_2).

1d34: use 1d32.

1d35: a separating sequence generates such $\mathcal{A}_1$.

1d37: recall 1a13.

1d38: (1d36); $\sigma(\varphi) = \sigma(A_1, A_2, \dots)$.

1d39: use 1d32 and (1d36).

1d40: use 1d39 and 1d29.

1d41: use 1d14.

1d42: use 1d41 and 1d38.

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