4 From random Borel functions to random closed sets

A random Borel set is not a random element of the Borel σ-algebra, but a random closed set is a random element of a standard Borel space of closed sets.

4a Random measurable maps

Indicators of random Borel sets are a special case of random Borel functions. The supremum of a random Borel function is measurable due to Sect. 3.

In Sect. 2c random functions were treated as random elements of some spaces of functions (see 2c10–2c13), much more restricted than all functions or all Borel functions. And indeed, the indicator of a random Borel set cannot be treated in this way (since the random Borel set cannot, recall Sect. 2d). Now we proceed in the spirit of 2d3. A probability space $(\Omega, \mathcal{F}, P)$ is assumed to be given, but rarely mentioned.[1]

4a1 Definition. A *random Borel function* is an equivalence class of maps $\Omega \to \mathbb{R}^{\mathbb{R}}$ that contains the map

$$\omega \mapsto \big(x \mapsto f(\omega, x) \big)$$

for some $(\mathcal{F} \times \mathcal{B}(\mathbb{R}))$-measurable $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

As before, "equivalent" means "differ on a negligible set only". As before, it is convenient to denote both objects by f; just denote $f(\omega) = f(\omega, \cdot) = \big(x \mapsto f(\omega, x) \big)$.

You may easily give an equivalent definition in the spirit of 2d2.

Compare it with 1d26–1d27 and 2c10–2c13.

[1] In Sect. 2 it was denoted by $(X, \mathcal{A}, \mu)$.

4a2 Remark. Why only Borel functions $\mathbb{R} \to \mathbb{R}$? In the spirit of 2d18, a *random measurable map* from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ is defined for any pair of measurable spaces, via a measurable function from $(\Omega \times X, \mathcal{F} \times \mathcal{A})$ to $(Y, \mathcal{B})$.

4a3 Remark. Random measurable maps $X \to \{0, 1\}$ are exactly the indicators of random measurable subsets of X.

4a4 Core exercise. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ be measurable spaces, f a random measurable map $X \to Y$, and g a random measurable map $Y \to Z$.[1] Then their composition h defined by

$$h(\omega) = g(\omega) \circ f(\omega)$$

(in other words, $h(\omega)(x) = g(\omega)\big(f(\omega)(x)\big)$, that is, $h(\omega, x) = g(\omega, f(\omega, x)))$ is a random measurable map $X \to Z$.

Prove it.

In particular, f (or g) may be nonrandom (not depending on ω).

Taking $Z = \{0, 1\}$ we see that the inverse image $f^{-1}(S)$ of a random measurable subset S of Y under a random measurable map f from X to Y is a random measurable subset of X. More formally,

$$f^{-1}(S) : \omega \mapsto f(\omega)^{-1}\big(S(\omega)\big) \,,$$

in other words, $f^{-1}(S)(\omega) = \{x : f(\omega)(x) \in S(\omega)\}$, that is, $f^{-1}(S) = \{(\omega, x) : \big(\omega, f(\omega, x)\big) \in S\}$.

In particular, a random Borel function leads to the random Borel sets $\{x : f(x) > 0\}$, $\{x : f(x) = 0\}$, $\{x : f(x) < 0\}$.

What about the supremum of a random Borel function,

$$\sup f : \omega \mapsto \sup f(\omega, \cdot) = \sup_x f(\omega, x) \,,$$

clearly an equivalence class of functions $\Omega \to (-\infty, +\infty]$, but what about measurability? Surprisingly, it follows from Theorem 3f23! (The same holds for infimum, of course.)

Recall 2c6: the distinction between $\mathcal{F}$-measurability and $\mathcal{F}_P$-measurability disappears on the level of equivalence classes.

4a5 Proposition. Let $(X, \mathcal{A})$ be a standard Borel space and f a random measurable function $X \to \mathbb{R}$. Then $\sup f$ is measurable.

[1] Both "random" objects use the same $(\Omega, \mathcal{F}, P)$, of course.

Proof. It is sufficient to prove that $\{\omega : \sup f(\omega, \cdot) > a\} \in \mathcal{F}_P$ for every $a \in \mathbb{R}$. To this end we apply Theorem 3f23 to the random Borel subset $f^{-1}\big((a,\infty)\big)$ of X. $\square$

Standardness matters...

4a6 Extra exercise. Let $(\Omega, \mathcal{F}, P)$ be $[0,1]$ with Lebesgue measure. Prove that for every (not just measurable) $Z \subset \Omega$ there exist a (not just standard) Borel space $(X, \mathcal{A})$ and a random measurable function $X \to \mathbb{R}$ such that $\sup f = \mathbf{1}_Z$.

4a7 Core exercise. Let $(X, \mathcal{A})$ be a standard Borel space and f a random Borel function $X \to \mathbb{R}$. Then the closure of the range of f,

$$S(\omega) = \operatorname{Closure}\big(\{f(\omega)(x) : x \in X\}\big) \subset \mathbb{R}$$

is a random Borel set.

Prove it.

4b Random continuous functions revisited

A random Borel function is not a random element, but a random continuous function is.

Now we have two ideas of a random continuous function, say, $[0,1] \to \mathbb{R}$. One idea: a random Borel function $[0,1] \to \mathbb{R}$ that happens to be continuous a.s. The other idea: a random element of the standard Borel space of continuous functions $[0,1] \to \mathbb{R}$ (recall 2c10(c)). The former is basically a measurable map $f : \Omega \times [0,1] \to \mathbb{R}$ such that $f(\omega, \cdot)$ is continuous for every x. The latter is basically a map $f : \Omega \times [0,1] \to \mathbb{R}$ such that $f(\omega, \cdot)$ is continuous for every ω, and $f(\cdot, x)$ is measurable for every x. Is it the same?

You see, joint measurability implies separate measurability (since the map $x \mapsto (x, y)$ is measurable), but the converse fails (try the graph of an arbitrary injection).[1]

4b1 Core exercise. Let X be a measurable space and $f_n : X \to [-\infty, +\infty]$ measurable functions.

(a) If $f : X \to [-\infty, +\infty]$, $f(\cdot) = \sup_n f_n(\cdot)$, then f is measurable.

(b) If $f : X \to [-\infty, +\infty]$, $f(\cdot) = \limsup_n f_n(\cdot)$, then f is measurable.

(c) If $f : X \to [-\infty, +\infty]$, $f_n(\cdot) \to f(\cdot)$ as $n \to \infty$, then f is measurable.

Prove it.

[1] On one hand, injections are a set of cardinality more than continuum. (Use a continuum of disjoint two-element sets.) On the other hand, analytic sets are a set of cardinality at most continuum, since they all are encoded by pairs (ρ, f) where ρ is a metric on $\{1, 2, 3, \dots\}$ and f is a ρ-continuous map $\{1, 2, 3, \dots\} \to \mathbb{R}$.

4b2 Lemma. Let $(\Omega, \mathcal{F})$ be a measurable space. If $f : \Omega \times [0,1] \to \mathbb{R}$ is such that $f(\omega, \cdot)$ is continuous for every $\omega \in \Omega$, and $f(\cdot, x)$ is measurable for every $x \in [0,1]$, then f is measurable.

Proof. $f(\cdot,\cdot) = \lim_n f_n(\cdot,\cdot)$ where $f_n(\omega, x) = f(\omega, \frac{k}{n})$ whenever $\frac{k}{n} \le x < \frac{k+1}{n}$. $\square$

4b3 Remark. We may take $\Omega = C[0,1]$ (the space of continuous functions) with the σ-algebra $\mathcal{F}$ generated by evaluations $g \mapsto g(x)$, and apply 4b2 to the map $(g,x) \mapsto g(x)$; interestingly,

$$g(x) \text{ is jointly measurable in } g \text{ and } x\,. \tag{4b4}$$

And conversely, (4b4) implies 4b2. Indeed, $\varphi : \omega \to f(\omega, \cdot)$ is a measurable map $\Omega \to C[0,1]$, therefore $(\omega, x) \mapsto \big(\varphi(\omega), x\big) \mapsto \varphi(\omega)(x) = f(\omega, x)$ is measurable.

4b5 Core exercise. Generalize 4b2, replacing $[0,1]$ with a separable metrizable space X.

4b6 Core exercise. The Borel σ-algebra of a separable metrizable space is generated by a sequence of continuous functions (real-valued).

Prove it.

4b7 Core exercise. Let X be a measurable space, Y a separable metrizable space, $f_n : X \to Y$ measurable functions, and $f_n(\cdot) \to f(\cdot)$ as $n \to \infty$, then f is measurable.

Prove it.

4b8 Core exercise. Generalize 4b5, replacing $\mathbb{R}$ with a separable metrizable space Y.

Thus, the two ideas of a random continuous map $X \to Y$ are equivalent (whenever X, Y are separable metrizable spaces).

4c Random semicontinuous functions

Unexpectedly, random semicontinuous functions are more tractable and useful than random continuous functions.

Indicator functions are rarely continuous (only for clopen sets). For closed sets, they are upper semicontinuous; for open sets — lower semicontinuous (see below).

4c1 Definition. Let X be a metrizable space. A function $f : X \to \mathbb{R}$ is *upper semicontinuous* if the set $\{x \in X : f(x) < a\}$ is open for every $a \in \mathbb{R}$; f is *lower semicontinuous* if the set $\{x \in X : f(x) > a\}$ is open for every $a \in \mathbb{R}$.

Equivalently one may require closeness of $\{x \in X : f(x) \ge a\}$ (upper) or $\{x \in X : f(x) \le a\}$ (lower).

A function is continuous if and only if it is both upper and lower semicontinuous.

Lemma 4b2, or equivalently (4b4), fails for upper semicontinuous functions. Here is why. Every measurable function of (f, x) is in fact a function of $(f|_A, x)$ for some countable A (recall 1d41). But $(f, x) \mapsto f(x)$ is not. Indeed, knowing that $f|_A(\cdot) = 0$ we still do not know $f(x)$ for $x \notin A$ (try $f = \mathbf{1}_{\{x\}}$).

Let us seek a better σ-algebra on upper semicontinuous functions.

4c2 Core exercise. Let $A \subset X$ be dense, and $f : X \to \mathbb{R}$ upper semicontinuous.

(a) $\inf_X f = \inf_A f$; prove it.

(b) It may happen that $\sup_X f > \sup_A f$; find an example.

4c3 Core exercise. Let $f : X \to \mathbb{R}$ be upper semicontinuous.

(a) $\inf_{\text{open } U \ni x} \sup_U f = f(x)$; prove it.

(b) It may happen that $\sup_{\text{open } U \ni x} \inf_U f < f(x)$; find an example.

4c4 Definition. Let X be a separable metrizable space. The measurable space $C_{\text{upper}}(X)$ is the set of all upper semicontinuous functions $X \to \mathbb{R}$ endowed with the σ-algebra generated by the functions $f \mapsto \sup_U f \in (-\infty, +\infty]$ where U runs over all open subsets of X.

By 4c3(a), evaluations $f \mapsto f(x)$ are measurable on $C_{\text{upper}}(X)$. But moreover...

4c5 Lemma. The map

$$C_{\text{upper}}(X) \times X \ni (f, x) \mapsto f(x) \in \mathbb{R}$$

is measurable.

Proof (sketch).

$$f(x) = \inf_{n : U_n \ni x} \sup_{U_n} f$$

where $(U_n)_n$ is a countable base of the topology. □

4c6 Core exercise. If $\varphi : \Omega \to C_{\text{upper}}(X)$ is measurable then $\varphi : \omega \mapsto f(\omega, \cdot)$ for some measurable $f : \Omega \times X \to \mathbb{R}$.

Prove it.

4c7 Core exercise. Let X be Polish, $f : \Omega \times X \to \mathbb{R}$ measurable, and $f(\omega, \cdot)$ upper semicontinuous for almost all ω. Then the map $\varphi : \omega \mapsto f(\omega, \cdot)$ is measurable from Ω to $C_{\text{upper}}(X)$.

Prove it.

We conclude: the two ideas of a random upper semicontinuous function on a Polish space are equivalent, as follows.

4c8 Proposition. For every Polish space X the formula

$$\varphi(\omega) = f(\omega, \cdot)$$

establishes a bijective correspondence between

(a) random measurable functions f from X to $\mathbb{R}$ such that $f(\omega, \cdot)$ is upper semicontinuous for almost all ω; and

(b) random elements φ of the measurable space $C_{\text{upper}}(X)$.

Given a set $K \subset \mathbb{R}$, we denote by $C_{\text{upper}}(X, K)$ the measurable subspace of $C_{\text{upper}}(X)$ consisting of all f such that $f(X) \subset K$.

4c9 Proposition. If X is a Polish space and $K \subset \mathbb{R}$ a compact set then $C_{\text{upper}}(X, K)$ is a standard Borel space.

Proof. We choose a compatible metric, and a countable base $\mathcal{E} \subset 2^X$ of the topology, consisting of bounded sets. The σ-algebra of $C_{\text{upper}}(X, K)$ is generated by the functions $f \mapsto \sup_U f$ for $U \in \mathcal{E}$ (think, why). Using these "coordinates" on $C_{\text{upper}}(X, K)$ (recall the paragraph after 2c7) we'll prove that the image B of $C_{\text{upper}}(X, K)$ in $K^{\mathcal{E}}$ is a Borel set.

If $h \in B$ then clearly
(a) $U \subset V$ implies $h(U) \le h(V)$; and
(b) $h(U) = \sup\{h(V) : V \in \mathcal{E}, \overline{V} \subset U, \operatorname{diam} V \le 0.5 \operatorname{diam} U\}$.

On the other hand, we'll prove that every $h \in K^{\mathcal{E}}$ satisfying (a) and (b) belongs to B (which implies easily that B is a Borel set). Given such h we define f by

$$f(x) = \inf\{h(U) : U \ni x\} ,$$

then $f \in C_{\text{upper}}(X, K)$ (think, why); we'll prove that h corresponds to f, that is,

$$h(U) = \sup_U f \quad \text{for all } U \in \mathcal{E} .$$

To this end, given $\varepsilon > 0$, first, using (b) we take $U_1 \in \mathcal{E}$ such that $\overline{U}_1 \subset U$, $\operatorname{diam} U_1 \le 0.5 \operatorname{diam} U$ and $h(U_1) \ge h(U) - 2^{-1}\varepsilon$. Second, we take $U_2 \in \mathcal{E}$ such that $\overline{U}_2 \subset U_1$, $\operatorname{diam} U_2 \le 0.5 \operatorname{diam} U_1$ and $h(U_2) \ge h(U_1) - 2^{-2}\varepsilon$. And so on.

We get $U \supset \overline{U}_1 \supset U_1 \supset \overline{U}_2 \supset U_2 \supset \dots$ and $\operatorname{diam} U_n \to 0$; by completeness, $U_1 \cap U_2 \cap \dots = \{x\}$ for some $x \in X$. Using (a) we get

$$f(x) = \inf_n h(U_n) \ge h(U) - \varepsilon$$

(think, why). Thus, $\sup_U f \ge h(U) - \varepsilon$ for all ε; therefore $\sup_U f = h(U)$ for all $U \in \mathcal{E}$, and $h \in B$. □

Following Sect. 4b we define $C(X)$ as the set of all continuous functions $X \to \mathbb{R}$, endowed with the σ-algebra generated by the evaluations $f \mapsto f(x)$ for $x \in X$.

4c10 Core exercise. $C(X)$ is a measurable subspace of $C_{\text{upper}}(X)$ (that is, their σ-algebras conform; it is not claimed to be a measurable subset).

Prove it.

Similarly, $C(X, K) \subset C_{\text{upper}}(X, K)$. We may also consider $K = [-\infty, +\infty]$ (evidently isomorphic to $K = [0, 1]$).

4c11 Remark. If X is compact then $C(X)$ is a standard Borel space. (This is a simple generalization of 2c10(c).)

4c12 Core exercise. If X is compact then $C(X)$ is a Borel subset of $C_{\text{upper}}(X)$.[1]

Prove it.

Strangely enough, for Polish X the space $C(X)$ is harder to investigate than $C_{\text{upper}}(X)$.[2]

For arbitrary $f : X \to \mathbb{R}$ we may define

$$f_*(x) = \sup_{U \ni x} \inf_U f \,, \quad f^*(x) = \inf_{U \ni x} \sup_U f \,,$$

where U runs over all open sets containing x (or equivalently, a basis of neighborhoods of x); $-\infty \le f_*(x) \le f(x) \le f^*(x) \le +\infty$. Note that f^* is

[1] In fact, Borel measurability follows from standardness by a general theorem.

[2] See Problem 7 on page 126 in: J.P.R. Christensen, "Topology and Borel structure", North-Holland 1974.

upper semicontinuous, and f_* is lower semicontinuous (think, why). Thus $f^* - f_*$ is upper semicontinuous, and the set

$$\{x : f_*(x) = f^*(x)\} = \bigcap_{\varepsilon>0} \{x : f^*(x) - f_*(x) < \varepsilon\}$$

of all continuity points of f is a G_δ-set (recall 1c14).

If $f \in C_{\text{upper}}(X)$ then $f^* = f$ by 4c3(a).

4c13 Core exercise. The map

$$C_{\text{upper}}(X) \times X \ni (f, x) \mapsto f_*(x) \in [-\infty, +\infty)$$

is measurable.

Prove it.

4c14 Core exercise. Let X be Polish, and φ a random upper semicontinuous function on X, then the set of all continuity points of $\varphi(\omega)$, treated as a map $\Omega \to 2^X$, is a random Borel subset of X.

Prove it.

By Theorem 3f23, the set $\{\omega : \varphi(\omega) \in C(X)\}$ is P-measurable.[1]

4d Random closed sets

The intersection of two random closed sets is more problematic than union, but still tractable.

Given a Polish space X, we treat the set $\mathbf{F}(X)$ of all closed subsets of X as a copy of $C_{\text{upper}}(X, \{0, 1\})$:

$$\mathbf{F}(X) \ni F \longleftrightarrow \mathbf{1}_F \in C_{\text{upper}}(X, \{0, 1\})\,. \tag{4d1}$$

Following 4c4 we endow $\mathbf{F}(X)$ with the σ-algebra generated by the sets

$$\{F \in \mathbf{F}(X) : F \cap U \ne \emptyset\} \tag{4d2}$$

where U runs over all open subsets of X. Lemma 4c5 and Propositions 4c8, 4c9 take the following form.

4d3 Lemma. The set $\{(F, x) : x \in F\}$ in $\mathbf{F}(X) \times X$ is Borel.

[1] In fact, all pairs (f, x) such that f is not continuous at x are a Borel subset of $C_{\text{upper}}(X) \times X$; its projection $C_{\text{upper}}(X) \setminus C(X)$ is analytic; thus, $C(X)$ is a *coanalytic* (and therefore universally measurable) subset of $C_{\text{upper}}(X)$. By a general theorem, $C(X)$ is analytic if and only if it is Borel.

4d4 Proposition. For every Polish space X, the following two conditions on an equivalence class S of maps $\Omega \to 2^X$ are equivalent:

(a) S is a random Borel set, closed almost surely;

(b) S is a random element of the Borel space $\mathbf{F}(X)$.

4d5 Proposition. $\mathbf{F}(X)$ is a standard Borel space (for every Polish space X).

This $\mathbf{F}(X)$ is basically the so-called Effros Borel space;[1] the only difference is that the empty set is excluded from the Effros Borel space (by Srivastava, but not by Kechris).

A result similar to Prop. 4d4 is available in descriptive set theory.[2] There Ω is treated as a standard Borel space $(\Omega, \mathcal{F})$ rather than probability space $(\Omega, \mathcal{F}, P)$, which makes the theory substantially harder; the Novikov separation theorem[3] is involved.

Prop. 4d5 is well-known,[4] with a harder proof based on an important work of Beer;[5] that proof uses 4d7 (below), but we deduce 4d7 from 4d5.

4d6 Core exercise. For every Polish space X the set of all singletons (that is, one-point sets) is a Borel subset of $\mathbf{F}(X)$, isomorphic (as a measurable space) to X.

Prove it.

Combining it with 4d5 and 2b11(a) we get the following.

4d7 Proposition. Every Polish space is a standard Borel space.

Let X be a Polish space and ρ a compatible metric on X.[6] We consider the distance between a point $x \in X$ and a set $F \subset X$:

$$\operatorname{dist}(x, F) = \inf\{\rho(x, y) : y \in F\}$$

($+\infty$ if $F = \emptyset$).

4d8 Core exercise. (a) For every $x \in X$ the function[7]

$$\mathbf{F}(X) \ni F \mapsto \operatorname{dist}(x, F) \in [0, \infty]$$

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[1]See Sect. 3.3 in: S.M. Srivastava, "A course on Borel sets", Springer 1998. Or Sect. 12.C in: A.S. Kechris, "Classical descriptive set theory", Springer 1995.

[2]See Kechris, Sect. 28.C, or Srivastava, Sect. 4.7.

[3]See Kechris, Sect. 28.B, or Srivastava, Sect. 4.6.

[4]Again, see Srivastava, Sect. 3.3, or Kechris, Sect. 12.C.

[5]G. Beer, "A Polish topology for the closed subsets of a Polish space", Proc. AMS 1991, **113**:4, 1123-1133.

[6]Note that (X, ρ) need not be complete.

[7]So-called distance functional.

is Borel measurable;

(b) these functions generate the σ-algebra of $\mathbf{F}(X)$;

(c) item (b) still holds when x runs over a dense subset of X.

Prove it.

Thus, $S : \Omega \to \mathbf{F}(X)$ is measurable if and only if $\operatorname{dist}\big(x, S(\cdot)\big)$ is measurable for all x.

4d9 Core exercise. The binary operation $(F_1, F_2) \mapsto F_1 \cup F_2$ on $\mathbf{F}(X)$ is Borel measurable. In other words, it is a Borel measurable map $\mathbf{F}(X) \times \mathbf{F}(X) \to \mathbf{F}(X)$.

Prove it.

Amazingly, the binary operation $(F_1, F_2) \mapsto F_1 \cap F_2$, in fact, is generally not Borel measurable! Moreover, the restriction map

$$\mathbf{F}(X) \ni F \mapsto F \cap Y \in \mathbf{F}(Y)$$

for a closed $Y \subset X$ is generally not measurable. Still more, the set $\{F : F \cap Y \ne \emptyset\}$ is generally not Borel measurable. In particular, this happens when X is a separable infinite-dimensional Hilbert space (say, l_2 or $L_2[0,1]$) and Y is its unit sphere.[1] The proof, based ultimately on Cantor's diagonal argument (recall Sect. 1c), is beyond our course.

4d10 Extra exercise. Let X be a Polish space and $Y \subset X$ a closed subset such that the restriction map $\mathbf{F}(X) \ni F \mapsto F \cap Y \in \mathbf{F}(Y)$ is not Borel measurable. Prove that the subset $C(Y, (0, \infty))$ of the space $C(Y, [0, \infty))$ is not Borel measurable.[2]

In a finite dimension the problem disappears (see 4d15(c)):

(4d11) binary operation $(F_1, F_2) \mapsto F_1 \cap F_2$ on $\mathbf{F}(\mathbb{R}^d)$ is Borel measurable.

4d12 Core exercise. Let X be a Polish space, ρ a compatible metric, and $Y \subset X$ a compact set. Then

(a) the function $F \mapsto \operatorname{dist}(F, Y) = \inf_{x \in F,\, y \in Y} \rho(x, y)$ is Borel measurable on $\mathbf{F}(X)$;

(b) the set $\{F : F \cap Y \ne \emptyset\}$ is Borel measurable in $\mathbf{F}(X)$;

(c) the restriction map $\mathbf{F}(X) \ni F \mapsto F \cap Y \in \mathbf{F}(Y)$ is Borel measurable.

Prove it.

[1] See Christensen, p. 77 or Kechris, Sect. 12.C and 27.B.

[2] Similarly, the subset $C(Y) = C(Y, (-\infty, +\infty))$ of the space $C(Y, [-\infty, +\infty])$ is not Borel measurable. By a general theorem, $C(Y)$ is not a standard Borel space for such Polish space Y. I do not know whether $C(Y, [-\infty, +\infty])$ is standard or just coanalytic.

Now recall 3a9.

4d13 Core exercise. Let X_1, X_2 be Polish spaces, then $(F_1, F_2) \mapsto F_1 \times F_2$ is a Borel measurable map $\mathbf{F}(X_1) \times \mathbf{F}(X_2) \to \mathbf{F}(X_1 \times X_2)$.

Prove it.

4d14 Core exercise. Let X be a compact metrizable space, then the binary operation $(F_1, F_2) \mapsto F_1 \cap F_2$ on $\mathbf{F}(X)$ is Borel measurable.

Prove it.

A metrizable space (or its subset) is called σ-compact[1] if it is the union of some sequence of compact sets.

4d15 Core exercise. (a) Generalize 4d12(c) to a σ-compact Y;

(b) generalize 4d14 to a σ-compact X;

(c) prove (4d11).

Let $S_1, S_2 : \Omega \to \mathbf{F}(X)$ be random closed sets, then $S_1 \cup S_2 : \omega \mapsto S_1(\omega) \cup S_2(\omega)$ is a random closed set by 4d9 (and 1d18, 1d4). What about $S_1 \cap S_2$? By 4d15(b), it is a random closed set provided that X is σ-compact. And nevertheless…

4d16 Lemma. Let X be a Polish space, $S_1, S_2, \dots : \Omega \to \mathbf{F}(X)$ random closed sets and $S : \omega \mapsto S_1(\omega) \cap S_2(\omega) \cap \dots$, then S is a random closed set.

Proof. Proposition 4d4 makes it evident! $\square$

What happens? Well, the map $(F_1, F_2) \mapsto F_1 \cap F_2$ need not be Borel measurable, but it is universally measurable.[2]

The set of all (closed) linear subspaces of a separable Hilbert space H is Borel measurable in $\mathbf{F}(H)$ (see below).

4d17 Lemma. Let X be a Polish space and $f : X \times X \to X$ a continuous map. Then all closed sets $F \subset X$ satisfying

$$\forall x, y \in F \quad f(x, y) \in F$$

are a Borel set in $\mathbf{F}(X)$.

Proof. We take a countable topological base $(U_n)_n$ of X and note that F does not belong to the examined set if and only if there exist k, l, m such that $f(U_k, U_l) \subset U_m$, $F \cap U_k \ne \emptyset$, $F \cap U_l \ne \emptyset$ and $F \cap U_m = \emptyset$. Here $f(U_k, U_l) = \{f(x, y) : x \in U_k, y \in U_l\}$. $\square$

[1]This notion was mentioned in Sect. 3b.

[2]Indeed, for open $U \subset X$ the set $\{(F_1, F_2) : F_1 \cap F_2 \cap U \ne \emptyset\} \subset \mathbf{F}(X) \times \mathbf{F}(X)$ is the projection of the Borel set $\{(F_1, F_2, x) : x \in F_1,\ x \in F_2,\ x \in U\} \subset \mathbf{F}(X) \times \mathbf{F}(X) \times X$.

This f is a binary operation; other operations (unary, ternary, ...) are treated similarly.

It remains to note that $F \in \mathbf{F}(H)$ is a linear subspace if and only if $x + y \in F$ for all $x, y \in F$ and $rx \in F$ for all $x \in F$ and rational r.

Thus, if needed, we may treat random linear subspaces. And even random σ-algebras (as follows).

4d18 Core exercise. Consider the measure algebra $\mathcal{A}/\overset{\mu}{\sim}$ of Lebesgue measure on $[0,1]$, endowed with the metric "dist" introduced in Sect. 2a. Then:

(a) $\mathcal{A}/\overset{\mu}{\sim}$ is Polish.

(b) All sub-σ-algebras of $\mathcal{A}/\overset{\mu}{\sim}$ are a Borel set in $\mathbf{F}(\mathcal{A}/\overset{\mu}{\sim})$.

Prove it.

Hints to exercises

4a4: apply 1d18 to $(\omega, x) \mapsto \big(\omega, f(\omega, x)\big)$.

4a7: think about supremum and infimum of $1/(f(\cdot) - r)$ for rational r.

4b5: take a dense sequence $(x_k)_k$ and consider the first k satisfying $\rho(x, x_k) \le \varepsilon$.

4b6: use a metric.

4b7: use 4b6 and 4b1(c).

4b8: use 4b7.

4c6: use 4c5 similarly to 4b3.

4c7: $\sup_U f(\omega, \cdot)$ is measurable by 4a5, since U is itself a standard Borel space.

4c10: use 4c2(a) and 4c3(a).

4c12: $\sup_U f - \inf_U f \to 0$ as $\operatorname{diam} U \to 0$.

4c13: similar to 4c5; and note 4c2(a).

4c14: $\omega \mapsto \varphi(\omega)^* - \varphi(\omega)_*$ is a random Borel function on X.

4d6: use a countable base of X.

4d8: think about $\{F : \operatorname{dist}(x, F) < r\}$.

4d9: just 4d2.

4d12: (a) use 4d8(a); (b) use (a); (c) given open $U \subset Y$, apply (b) to compact subsets of U.

4d13: use 4d8; or, alternatively, take countable bases and consider $U_n \times V_n$.

4d14: intersect $F_1 \times F_2$ with the diagonal $\{(x, x) : x \in X\}$.

4d15: (a) an open subset of Y is σ-compact.

4d18: (a) completeness is ensured by 3a10(d); deduce separability from 2d16(c). (b) a closed algebra is a σ-algebra.

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