2 The Lebesgue measure

Lebesgue measure on $\mathbb{R}^d$ is constructed. It turns $\mathbb{R}^d$ into a measure space.

2a Jordan measure

Jordan measure on $\mathbb{R}^d$ (called also Jordan content) is closely related to the d-dimensional Riemann integral. Both are treated in the course "Analysis 3". I borrow from that course several facts listed below. See also Sect. 1.1.2 "Jordan measure" in the textbook by Tao.

2a1 Fact. A set $E \subset \mathbb{R}^d$ is Jordan measurable (in other words, a Jordan set) if and only if its indicator function $\mathbb{1}_E$ is Riemann integrable; in this case the Jordan measure of E is the Riemann integral,

$$m(E) = \int_{\mathbb{R}^d} \mathbb{1}_E \, .$$

Clearly, E must be bounded, and $m(E) \in [0, \infty)$.

2a2 Fact. If $(a_1, b_1) \times \dots \times (a_d, b_d) \subset E \subset [a_1, b_1] \times \dots \times [a_d, b_d]$, then E is Jordan, and $m(E) = (b_1 - a_1) \cdot \dots \cdot (b_d - a_d)$.

2a3 Fact. If E, F are Jordan, then $E \cup F$, $E \cap F$ and $E \setminus F$ are Jordan; and if $E \cap F = \emptyset$, then

$$m(E \cup F) = m(E) + m(F) \, . \qquad \text{(additivity)}$$

Clearly, $m(E \cup F) + m(E \cap F) = m(E) + m(F)$, and $m(E \cup F) \le m(E) + m(F)$ (subadditivity). Also, $E \subset F$ implies $m(E) \le m(F)$ (monotonicity).

2a4 Fact (regularity)**.** For every Jordan set E and every $\varepsilon > 0$ there exist Jordan sets K, U such that K is compact, U is open, $K \subset E \subset U$, and $m(U \setminus K) \le \varepsilon$.[1]

2a5 Fact. Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transformation, and $b \in \mathbb{R}^d$. Then for every Jordan set $E \subset \mathbb{R}^d$ the set $LE+b = \{Lx+b : x \in E\}$ is Jordan, and

$$m(LE+b) = |\det L| m(E) \, .$$

In particular, the Jordan measure is invariant under shifts, rotations and reflections.

The following result is of little interest to Riemann integration, but crutial for Lebesgue integration.

2a6 Proposition. Let $E, E_1, E_2, \dots \subset \mathbb{R}^d$ be Jordan sets. If $E \subset \cup_i E_i$, then $m(E) \le \sum_i m(E_i)$.

Proof. It is sufficient to prove that $m(E) \le 2\varepsilon + \sum_i m(E_i)$ for arbitrary $\varepsilon > 0$. Given ε, we take $\varepsilon_1, \varepsilon_2, \dots > 0$ such that $\varepsilon_1 + \varepsilon_2 + \dots \le \varepsilon$ (for instance, $\varepsilon_i = 2^{-i}\varepsilon$), open Jordan $U_i \supset E_i$ such that $m(U_i) \le m(E_i) + \varepsilon_i$, and a compact Jordan set $K \subset E$ such that $m(K) \ge m(E) - \varepsilon$.

We have $K \subset \cup_i U_i$; by compactness, there exists i such that $K \subset U_1 \cup \dots \cup U_i$. Thus, $m(E) \le \varepsilon + m(K) \le \varepsilon + m(U_1) + \dots + m(U_i) \le 2\varepsilon + m(E_1) + \dots + m(E_i)$. $\square$

2a7 Corollary. Let $E, E_1, E_2, \dots \subset \mathbb{R}^d$ be Jordan sets. If $E = \uplus_i E_i$,[2] then $m(E) = \sum_i m(E_i)$.

2b Open sets, compact sets; outer measure, inner measure[3]

2b1 Definition. *Lebesgue measure of an open set* $U \subset \mathbb{R}^d$ is its inner Jordan measure:[4]

$$m(U) = \sup\{m(E) : \text{Jordan } E \subset U\} \in [0, \infty] \, .$$

The notation is consistent: if U is Jordan, then this supremum is equal to the Jordan measure of U.

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[1] A stronger formulation $K \subset E^\circ \subset E \subset \overline{E} \subset U$ holds, but we do not need it.

[2] It means, $E_i \cap E_j = \emptyset$ for $i \ne j$, and $E = \cup_i E_i$.

[3] Our 2b–2d follow stages 3–6 of Sect. 2A in the textbook by Jones. About Carathéodory, see Remark on p. 55 there: "But I believe the slow and deliberate development we have given is preferable for the beginner."

[4] Recall Sect. 1d: for an open set, its *inner* Jordan measure is relevant.

2b2 Exercise. Let $U \subset \mathbb{R}^d$ be an open set, and $E, E_1, E_2, \dots \subset \mathbb{R}^d$ Jordan sets.

(a) If $U \subset \cup_i E_i$, then $m(U) \le \sum_i m(E_i)$.

(b) If $U = \uplus_i E_i$, then $m(U) = \sum_i m(E_i)$.

Prove it.

2b3 Exercise. Every open set $U \subset \mathbb{R}^d$ is $\uplus_i E_i$ for some Jordan sets $E_1, E_2, \dots \subset \mathbb{R}^d$.

Prove it.[1,2]

2b4 Corollary (subadditivity)**.** $m(U \cup V) \le m(U) + m(V)$ for all open $U, V \subset \mathbb{R}^d$.

2b5 Lemma (monotone convergence for open sets)**.** Let $U, U_1, U_2, \dots \subset \mathbb{R}^d$ be open sets. If $U_i \uparrow U$,[3] then $m(U_i) \uparrow m(U) \in [0, \infty]$.

Proof. Clearly, $m(U_1) \le m(U_2) \le \dots \le m(U)$, therefore $\lim_i m(U_i) \le m(U)$. It is sufficient to prove that $\lim_i m(U_i) > a$ for arbitrary $a < m(U)$.

Given $a < m(U) = \sup\{m(E) : \text{Jordan } E \subset U\}$, we take a Jordan $E \subset U$ such that $m(E) > a$. Using 2a4 we take a compact Jordan $K \subset E$ such that $m(K) > a$. By compactness, there exists i such that $K \subset U_i$. Thus, $a < m(K) \le m(U_i) \le \lim_j m(U_j)$. □

Countable subadditivity follows:[4]

$$m(U_1 \cup U_2 \cup \dots) \le m(U_1) + m(U_2) + \dots \quad \text{for all open sets } U_1, U_2, \dots \subset \mathbb{R}^d .$$

2b6 Definition. *Outer measure* $m^*(A)$ of a set $A \subset \mathbb{R}^d$ is

$$m^*(A) = \inf\{m(U) : \text{open } U \supset A\} .$$

Clearly, $m^*(U) = m(U)$ for open U.

2b7 Exercise (countable subadditivity)**.**

$$m^*(A_1 \cup A_2 \cup \dots) \le m^*(A_1) + m^*(A_2) + \dots \quad \text{for all } A_1, A_2, \dots \subset \mathbb{R}^d .$$

Prove it.[5]

---

[1] Hint: try cubes of the form $[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}) \times \dots \times [\frac{i_d}{2^n}, \frac{i_d+1}{2^n})$.

[2] Tao, Lemma 1.2.11.

[3] It means, $U_1 \subset U_2 \subset \dots$ and $U = \cup_i U_i$.

[4] Since $U_1 \cup \dots \cup U_i \uparrow U_1 \cup U_2 \cup \dots$, and $m(U_1 \cup \dots \cup U_i) \le m(U_1) + \dots + m(U_i)$. Alternatively, the argument of 2b4 may be generalized.

[5] Hint: $\varepsilon_1 + \varepsilon_2 + \dots \le \varepsilon$.

2b8 Definition. A set $Z \subset \mathbb{R}^d$ is a *null set* if $m^*(Z) = 0$.

Every subset of a null set is null.
A Jordan set of zero Jordan measure is null (due to 2a4).
Countable union of null sets is a null set (by countable subadditivity).

2b9 Definition. *Lebesgue measure of a compact set* $K \subset \mathbb{R}^d$ is its outer Jordan measure:[1]

$$m(K) = \inf\{m(E) : \text{Jordan } E \supset K\} .$$

The notation is consistent: if K is Jordan, then this infimum is equal to the Jordan measure of K.

Subadditivity for compact sets, $m(K_1 \cup K_2) \le m(K_1) + m(K_2)$, follows readily from subadditivity for Jordan sets.

2b10 Exercise. If K is compact, U is open, and $K \subset U$, then
(a) there exists a Jordan set E such that $K \subset E \subset U$;
(b) $m(K) \le m(U)$;
(c) and moreover, $m(K) < m(U)$.
Prove it.[2]

2b11 Exercise. If K, L are compact and $K \cap L = \emptyset$, then
(a) there exist Jordan sets E, F such that $K \subset E$, $L \subset F$, and $E \cap F = \emptyset$;
(b) $m(K \uplus L) = m(K) + m(L)$.
Prove it.

2b12 Definition. *Inner measure* $m_*(A)$ of a set $A \subset \mathbb{R}^d$ is

$$m_*(A) = \sup\{m(K) : \text{compact } K \subset A\} .$$

Clearly, $m_*(K) = m(K)$ for compact K. Also, $m_*(A) \le m^*(A)$ due to 2b10(b).

2b13 Exercise (superadditivity)**.**
(a) $m_*(A \uplus B) \ge m_*(A) + m_*(B)$ whenever $A \cap B = \emptyset$;
(b) $m_*(A_1 \uplus A_2 \uplus \dots) \ge m_*(A_1) + m_*(A_2) + \dots$ whenever A_i are pairwise disjoint.
Prove it.

2b14 Lemma (regularity)**.**
$m_*(U) = m(U)$ for open U;
$m^*(K) = m(K)$ for compact K.

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[1] Recall Sect. 1d: for a compact set, its *outer* Jordan measure is relevant.
[2] Hint: $\operatorname{dist}(K, \mathbb{R}^d \setminus U) > 0$; try a finite union of small cubes.

Proof. First, $m_*(U) \le m(U)$ by 2b10(b). Second, given $c < m(U)$, we take Jordan $E \subset U$ such that $m(E) > c$ by 2b1, and compact $K \subset E$ such that $m(K) > c$ by 2a4. Thus, $m_*(U) = m(U)$.

For K, the argument is similar: 2b10(b) again, 2b9, and the other part of 2a4. □

2c Measurable sets of finite measure

2c1 Definition. A set $A \subset \mathbb{R}^d$ is *integrable*[1] if $m_*(A) = m^*(A) < \infty$; in this case its (Lebesgue) measure is

$$m(A) = m_*(A) = m^*(A) \, .$$

Open sets of finite measure, as well as compact sets, are integrable by 2b14, and the notation is consistent (the same $m(A)$ as before).

2c2 Lemma (additivity)**.** If A, B are integrable and $A \cap B = \emptyset$, then $A \uplus B$ is integrable and $m(A \uplus B) = m(A) + m(B)$.

Proof. By 2b7 and 2b13,

$$\begin{aligned} m^*(A \uplus B) \le m^*(A) + m^*(B) = m(A) + m(B) = \\ = m_*(A) + m_*(B) \le m_*(A \uplus B) \le m^*(A \uplus B) \, , \end{aligned}$$

therefore they all are equal. □

In particular, $m(U) = m(K) + m(U \setminus K)$ whenever U is open, K is compact, and $K \subset U$.

2c3 Exercise (sandwich)**.** A set $A \subset \mathbb{R}^d$ is integrable if and only if for every $\varepsilon > 0$ there exist open U and compact K such that $K \subset A \subset U$ and $m(U \setminus K) \le \varepsilon$.

Prove it.

2c4 Lemma. If A, B are integrable, then $A \cup B$, $A \cap B$ and $A \setminus B$ are integrable.

Proof. Given $\varepsilon > 0$, we take compact K, L and open U, V such that $K \subset A \subset U$, $L \subset B \subset V$, $m(U \setminus K) \le \varepsilon$ and $m(V \setminus L) \le \varepsilon$. We get a sandwich for $A \setminus B$ as follows:

$$\underbrace{K \setminus V}_{\text{compact}} \subset A \setminus B \subset \underbrace{U \setminus L}_{\text{open}} \, .$$

[1]Not a standard terminology. Just a shortcut for "measurable set of finite measure". Equivalent to integrability of $\mathbb{1}_A$.

We note that $(U \setminus L) \setminus (K \setminus V) \subset (U \setminus K) \cup (V \setminus L)$, therefore $m\big((U \setminus L) \setminus (K \setminus V)\big) \le 2\varepsilon$ by 2b4, which proves integrability of $A \setminus B$.

Integrability of $A \cap B = A \setminus (A \setminus B)$ and $A \cup B = (A \setminus B) \uplus B$ follows by 2c2. $\square$

2c5 Exercise. Prove integrability of $A \cap B$ and of $A \cup B$ using sandwich and not using integrability of $A \setminus B$.

2c6 Proposition. Let sets $A_1, A_2, \dots \subset \mathbb{R}^d$ be integrable, and $A = A_1 \cup A_2 \cup \dots$ satisfy $m^*(A) < \infty$. Then A is integrable, and $m(A) \le m(A_1) + m(A_2) + \dots$ If in addition A_i are (pairwise) disjoint, then $m(A) = m(A_1) + m(A_2) + \dots$

Proof. We start with the disjoint case: $A = \uplus_i A_i$. By 2b7 and 2b13(b),

$$m^*(A) \le \sum_i m^*(A_i) = \sum_i m(A_i) = \sum_i m_*(A_i) \le m_*(A) \le m^*(A)\,,$$

therefore they all are equal, which shows that A is integrable and $m(A) = \sum_i m(A_i)$.

In the general case we introduce disjoint sets B_i with the same union as follows:

$$B_1 = A_1\,, \quad B_2 = A_2 \setminus A_1\,, \quad B_3 = A_3 \setminus (A_1 \cup A_2)\,, \dots$$

then $\uplus_i B_i = A$. By 2c4, B_i are integrable. Thus, A is integrable, and $m(A) = \sum_i m(B_i) \le \sum_i m(A_i)$. $\square$

2d Measurable sets in general

2d1 Definition. A set $A \subset \mathbb{R}^d$ is *measurable* if for every integrable set C, the set $A \cap C$ is integrable; in this case the measure of A is

$$m(A) = \sup\{ m(A \cap C) : \text{integrable } C \}\,.$$

If A is integrable, then it is measurable (by 2c4), and the notation is consistent: this supremum is equal to $m(A)$ defined earlier.

2d2 Lemma. If A is measurable and $m^*(A) < \infty$, then A is integrable.

Proof. We take integrable $C_1, C_2, \dots$ (for instance, cubes) such that $\cup_i C_i = \mathbb{R}^d$ and apply 2c6 to $A = (A \cap C_1) \cup (A \cap C_2) \cup \dots$ $\square$

2d3 Proposition (measurable sets are an algebra of sets)**.** If $A, B \subset \mathbb{R}^d$ are measurable, then $A \cup B$, $A \cap B$ and $\mathbb{R}^d \setminus A$ are measurable.

Proof. For integrable C the set $(\mathbb{R}^d \setminus A) \cap C = C \setminus (A \cap C)$ is integrable (by 2c4); thus, $\mathbb{R}^d \setminus A$ is measurable.

Similarly, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ is integrable, therefore $A \cup B$ is measurable.

For $A \cap B$ use the same argument, or take the complement. $\square$

2d4 Proposition (measurable sets are a σ-algebra)**.** If $A_1, A_2, \dots \subset \mathbb{R}^d$ are measurable, then $A_1 \cup A_2 \cup \dots$ and $A_1 \cap A_2 \cap \dots$ are measurable.

Proof. For integrable C the set $(\cup_i A_i) \cap C = \cup_i (A_i \cap C)$ is integrable by 2c6, thus $\cup_i A_i$ is measurable.

For the intersection use the same argument, or take the complement. $\square$

2d5 Proposition (countable additivity)**.** If $A_1, A_2, \dots \subset \mathbb{R}^d$ are measurable and (pairwise) disjoint, then

$$m(A_1 \uplus A_2 \uplus \dots) = m(A_1) + m(A_2) + \dots \in [0, \infty] \, .$$

Proof. Denote $A = \uplus_i A_i$. For integrable C, by 2c6, $m(A \cap C) = \sum_i m(A_i \cap C)$. We have $\sup_C m(A \cap C) = m(A)$ and $\sup_C m(A_i \cap C) = m(A_i)$; it is sufficient to prove that $\sup_C \sum_i m(A_i \cap C) \ge \sum_i m(A_i)$ (indeed, "$\le$" is trivial).

We assume that $m(A_i) < \infty$ for all i (otherwise the claim is trivial). Given n and $\varepsilon > 0$, we take integrable $C_1, \dots, C_n$ such that $m(A_i \cap C_i) \ge m(A_i) - \frac{\varepsilon}{n}$ for $i = 1, \dots, n$, then $\sum_{i=1}^n m(A_i \cap C) \ge \sum_{i=1}^n m(A_i) - \varepsilon$ where $C = C_1 \cup \dots \cup C_n$. Thus, $\sup_C \sum_{i=1}^n m(A_i \cap C) \ge \sum_{i=1}^n m(A_i)$ for all n. $\square$

Additivity is a special case: $m(A \uplus B) = m(A) + m(B) \in [0, \infty]$.

2d6 Proposition. All open sets and all closed sets are measurable.

Proof. We take integrable compact sets $C_1, C_2, \dots$ (for instance, cubes) such that $\cup_i C_i = \mathbb{R}^d$. For a closed F, compact sets $F \cap C_i$ are integrable, therefore measurable, hence $F = \cup_i (F \cap C_i)$ is measurable.

For open set, take the complement. $\square$

2d7 Remark (regularity)**.** For every measurable A,

$$\sup_{\text{compact } K \subset A} m(K) \;=\; m(A) \;=\; \inf_{\text{open } U \supset A} m(U) \, .$$

Proof. The left equality: $m(A) = \sup\{m(C) : \text{integrable } C \subset A\}$ by 2d1, where $m(C) = m_*(C) = \sup\{m(K) : \text{compact } K \subset C\}$ by 2c1 and 2b12.

The right equality is trivial when $m(A) = \infty$; otherwise A is integrable by 2d2, and $m(A) = m^*(A) = \inf\{m(U) : \text{open } U \supset A\}$ by 2c1 and 2b6.

2d8 Exercise. Let us define a zigzag sandwich[1] (of Jordan sets) as consisting of Jordan sets $E_{k,l}$, $F_{k,l}$ and (generally not Jordan) sets E_k, F_k, E, F such that $E_{k,l} \downarrow E_k$ (as $l \to \infty$) and $F_{k,l} \uparrow F_k$ for every k, and $E_k \uparrow E$, $F_k \downarrow F$. Prove that[2]

(a) A set $A \subset \mathbb{R}^d$ is integrable if and only if there exists a zigzag sandwich such that $E \subset A \subset F$ and

$$\lim_k \lim_l m(E_{k,l}) = \lim_k \lim_l m(F_{k,l}) < \infty ;$$

and in this case

$$m(A) = \lim_k \lim_l m(E_{k,l}) = \lim_k \lim_l m(F_{k,l}) .$$

(b) A set $A \subset \mathbb{R}^d$ is measurable if and only if there exists a zigzag sandwich such that $E \subset A \subset F$ and $F \setminus E$ is a null set; and in this case[3]

$$m(A) = m(E) = m(F) \in [0, \infty] .$$

2e Measure space

2e1 Definition. Let X be a set, and S some set of subsets of X (that is, $S \subset 2^X$).
(a) S is an *algebra* of sets,[4] if[5]

$$\emptyset, X \in S ; \quad \forall A, B \in S \;\; A \cup B, A \cap B, X \setminus A \in S ;$$

(b) S is a σ*-algebra* (in other words, σ-field), if S is an algebra of sets, and

$$\forall A_1, A_2, \dots \in S \;\; (\cup_i A_i), (\cap_i A_i) \in S ;$$

(c) if S is a σ-algebra on X, then the pair (X, S) is called a *measurable space.*

2e2 Definition. (a) A *measure*[6] on a measurable space (X, S) is a function $\mu : S \to [0, \infty]$ such that $\mu(\emptyset) = 0$, and

$$\mu(A \uplus B) = \mu(A) + \mu(B) \qquad \text{(additivity)}$$

[1] This is the zigzag sandwich in the sense of Sect. 1e, but for sets rather than functions.

[2] Hint: 2d7, 2b9, 2b1.

[3] Do you think that in this case $m(A) = \lim_k \lim_l m(E_{k,l}) = \lim_k \lim_l m(F_{k,l})$?

[4] Called also a concrete Boolean algebra.

[5] Surely you can shorten this (and following) definition(s)...

[6] Ridiculously, "probability measures", "nonatomic measures", "finite measures" etc. are (special cases of) measures, but "signed measures", "complex measures", "vector measures", "finitely additive measures" etc. are not; rather, they are generalized measures.

whenever $A, B \in S$ are disjoint; and

$$\mu(\uplus_i A_i) = \sum_i \mu(A_i) \qquad \text{(countable additivity)}$$

whenever $A_1, A_2, \dots \in S$ are (pairwise) disjoint;

(b) if μ is a measure on (X, S), then the triple (X, S, μ) is called a *measure space.*

2e3 Example. All Jordan sets in $\mathbb{R}^d$ together with their complements are an algebra of sets, but not a σ-algebra.

2e4 Example. All (Lebesgue) measurable sets in $\mathbb{R}^d$ are a σ-algebra; it turns $\mathbb{R}^d$ into a measurable space. The Lebesgue measure is a measure on this measurable space, and turns it into a measure space.

2f Rotation invariance

2f1 Proposition. Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transformation, and $b \in \mathbb{R}^d$. Then for every $A \subset \mathbb{R}^d$, A is measurable if and only if the set $LA + b = \{Lx + b : x \in A\}$ is measurable, and in this case

$$m(LA + b) = |\det L| m(A) \,.$$

Proof. We denote $Lx + b$ by Tx.

First, let A be integrable. We take a zigzag sandwich for A according to 2d8(a). By 2a5, sets $T(E_{k,l})$, $T(F_{k,l})$ are Jordan, and $m\big(T(E_{k,l})\big) = |\det L| m(E_{k,l})$, $m\big(T(F_{k,l})\big) = |\det L| m(F_{k,l})$. We have $T(E_{k,l}) \downarrow T(E_k)$ and $T(F_{k,l}) \uparrow T(F_k)$ (since T is a bijection); also, $T(E_k) \uparrow T(E)$, $T(F_k) \downarrow T(F)$, and $T(E) \subset T(A) \subset T(F)$. We get a zigzag sandwich for $T(A)$; thus, $T(A)$ is integrable, and $m(T(A)) = |\det L| m(A)$. The same holds for $T^{-1} : y \mapsto L^{-1}y - L^{-1}b$, thus, A is integrable if and only if $T(A)$ is integrable.

Now, let A be measurable. It means that $A \cap C$ is integrable for all integrable C. Thus, $T(A) \cap T(C) = T(A \cap C)$ is integrable for all C such that $T(C)$ is integrable. It means that $T(A)$ is measurable. The same applies to T^{-1}. Finally, $m(A) = \sup\{m(A \cap C) : \text{integrable } C\} = (1/|\det L|) \sup\{m(T(A) \cap T(C)) : \text{integrable } T(C)\} = (1/|\det L|) m(T(A))$. $\square$

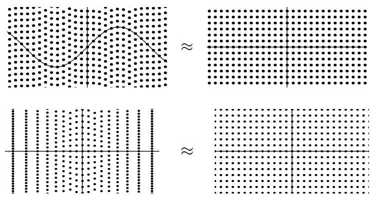
2f2 Corollary. [1] (a) The Lebesgue measure is well-defined in every d-dimensional Euclidean space.

[1] The same applies to affine spaces.

Indeed, every orthonormal basis in such space $\mathcal{E}$ leads to a linear isometry $L : \mathcal{E} \to \mathbb{R}^d$; we take $m(A) = m(L(A))$; by 2f1, the result does not depend on the basis.

(b) The Lebesgue σ-algebra is well-defined in every d-dimensional vector space, and the Lebesgue measure (on such space) is defined up to a coefficient.

2f3 Remark.
Prop. 2f1 generalizes readily to nonlinear bijections $T : \mathbb{R}^d \to \mathbb{R}^d$; if T preserves the Jordan measure, then T preserves the Lebesgue measure. Recall examples of nonlinear measure preserving transformations from Sect. 1b.



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