3 Measurable map, pushforward measure

Various σ-algebras, measures and maps are interrelated.

3a Introduction

Here are some definitions, and related facts that will be proved soon.

3a1 Definition. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *measurable* if it satisfies the following equivalent conditions:

(a) for every $t \in R$ the set $\{x : f(x) \le t\} \subset \mathbb{R}^d$ is measurable;

(b) for every interval $I \subset R$ the set $f^{-1}(I) \subset \mathbb{R}^d$ is measurable;

(c) for every open set $U \subset R$ the set $f^{-1}(U) \subset \mathbb{R}^d$ is measurable.

More generally:

3a2 Definition. Given a measurable space (X, S), a map $f : X \to \mathbb{R}^n$, $f(x) = \big(f_1(x), \dots, f_n(x)\big)$, is called *measurable* if it satisfies the following equivalent conditions:

(a) for every $t = (t_1, \dots, t_n) \in R^n$ the set $\{x : f_1(x) \le t_1, \dots, f_n(x) \le t_n\} \subset X$ belongs to the σ-algebra S;

(b) for every box $I \subset R^n$ the set $f^{-1}(I) \subset X$ belongs to the σ-algebra S;

(c) for every open set $U \subset R^n$ the set $f^{-1}(U) \subset X$ belongs to the σ-algebra S.

Also, a map $f : X \to \mathbb{R}^n$ is measurable if and only if its coordinate functions $f_1, \dots, f_n$ are measurable.

With respect to measurability, complex-valued functions $X \to \mathbb{C}$ may be treated as just $X \to \mathbb{R}^2$ (and $X \to \mathbb{C}^n$ as $X \to \mathbb{R}^{2n}$).

Warning: measurability of $f : X \to \mathbb{R}^n$ does not imply measurability of the set $f^{-1}(A)$ for arbitrary measurable sets $A \subset \mathbb{R}^n$. Not even in the simplest case, when $n = 1$ and (X, S) is $\mathbb{R}$ with the Lebesgue σ-algebra.

Rather, measurability of $f : X \to \mathbb{R}^n$ is equivalent to measurability of the set $f^{-1}(B)$ for every Borel set $B \subset \mathbb{R}^n$, that is, a set B that belongs to the Borel σ-algebra treated below. And a measurable f leads to a pushforward measure $f_* m : B \mapsto m(f^{-1}(B))$ on the Borel σ-algebra (rather than Lebesgue σ-algebra).

3b Borel sets

First, recall such notions as the linear span of a set of vectors in a vector space. In other words: the subspace generated by the given subset. It may be defined "via sets" as the least subspace that contains the given set. Or, equivalently, "via elements" as consisting of all linear combinations of the given elements.

Another example: the subgroup generated by a subset of a group. "Via sets" it is the least subgroup that contains the given set. "Via elements" it consists of $x_1^{k_1} \dots x_n^{k_n}$ for all n, all $x_1, \dots, x_n$ of the given set, and all $k_1, \dots, k_n \in \mathbb{Z}$.

Definitions "via elements" reveal the cardinality of the generated set; definitions "via sets" do not. For example: the linear span of a set of cardinality continuum is a set of cardinality continuum. Also, the subgroup generated by a countable set is countable.

Likewise, in the algebra of all subsets of a set X we may consider the (sub)algebra generated by a set (of subsets of X). For example: the algebra (of sets) generated by all intervals; here $X = \mathbb{R}$. (Can you give an equivalent definition "via elements"?)

Now, what about the generated σ-algebra? This is a harder matter, because the relevant operations are not at all binary; they take infinitely many arguments!

Still, no problem with the definition "via sets": just the least σ-algebra that contains all given sets.[1] Its existence is easy to see: the intersection of σ-algebras is always a σ-algebra (no matter how many σ-algebras are intersected), just as the intersection of vector spaces is always a vector space; infinitely many arguments are harmless at this point. And, frankly, this is enough for the theory. But hardly enough for our intuition; it is rather unnatural, to use Borel sets while having only a slight idea, what they really are and how many Borel sets exist.

[1]Well, no problem within the set theory. However, this definition is terribly impredicative: it makes sense only if we agree that not only all subsets of X, but also all sets of subsets of X exist in some Platonic reality, independently of out intellectual activity! If you are interested, what is it about, see for instance "Induction and predicativity" by Peter Smith (find item 20 in the list there).

Here is an equivalent definition "via elements" for the algebra S (not σ-algebra yet) generated by a given set G of subsets of X (call them "generators"): S consists of all sets of the form

$$\bigcup_{k=1}^{m} \bigcap_{l=1}^{n} A_{k,l}$$

where each $A_{k,l}$ is a generator or its complement, that is, $\forall k,l \ \big(A_{k,l} \in G \vee (X \setminus A_{k,l}) \in G \big)$.

Clearly, $A, B \in S \implies (A \cup B) \in S$. What about $A \cap B$? It belongs to S due to distributivity: $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$.

This argument fails badly for countable operations; trying to open the brackets in $(A_1 \cup B_1) \cap (A_2 \cup B_2) \cap \dots$ we get the union of a continuum of intersections!

One zigzag is enough for generating an algebra of sets, but not enough for σ-algebra.[1]

So, how many zigzags are needed? By n zigzags we get

$$\bigcup_{k_1=1}^{\infty} \bigcap_{l_1=1}^{\infty} \dots \bigcup_{k_n=1}^{\infty} \bigcap_{l_n=1}^{\infty} A_{k_1,l_1,\dots,k_n,l_n} \, .$$

But what about $A = A_1 \cap A_2 \cap \dots$ where A_n is obtained by n zigzags? This is

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k_1=1}^{\infty} \bigcap_{l_1=1}^{\infty} \dots \bigcup_{k_n=1}^{\infty} \bigcap_{l_n=1}^{\infty} A_{n,k_1,l_1,\dots,k_n,l_n} \, .$$

Note that the number of indices in $A_{n,k_1,l_1,\dots,k_n,l_n}$ is not fixed, it depends on the first index. This is instructive. The set A is encoded by the subset

$$T = \{(i_1, \dots, i_n) : n \le 2i_1 + 1\}$$

of the set $\{1, 2, \dots\}^{<\infty}$ of all finite sequences of natural numbers, and a function $(i_1, \dots, i_n) \mapsto A_{i_1,\dots,i_n}$ on a subset $T_0 = \{(i_1, \dots, i_n) : n = 2i_1 + 1\}$ of T.

Generally, we consider a tree $T \subset \{1, 2, \dots\}^{<\infty}$; here "tree" means that

$$(i_1, \dots, i_n) \in T \implies (i_1, \dots, i_{n-1}) \in T \, .$$

The empty sequence (of length 0) belongs to T ("the root of the tree"). The tree T is required to be well-founded; it means no infinite branch, that is,

$$\neg \, \exists i_1, i_2, \dots \ \forall n \ \ (i_1, \dots, i_n) \in T \, .$$

[1] See also the last footnote in Sect. 1e.

A vertex $(i_1, \dots, i_n) \in T$ is called a leaf if $\forall i_{n+1} \; (i_1, \dots, i_n, i_{n+1}) \notin T$; we denote by T_0 the set of all leaves of the tree T.

A Borel code consists, by definition, of a well-founded tree $T \subset \{1, 2, \dots\}^{<\infty}$ and a function $(i_1, \dots, i_n) \mapsto A_{i_1,\dots,i_n}$ on T_0 such that each $A_{i_1,\dots,i_n}$ is either a generator or its complement. This notion is an appropriate formalization of the idea of an infinite (but countable) formula.

Given a Borel code, there exists one and only one function $(i_1, \dots, i_n) \mapsto B_{i_1,\dots,i_n}$ on T (whose values are subsets of X) such that

$$B_{i_1,\dots,i_n} = \begin{cases} A_{i_1,\dots,i_n} & \text{if } (i_1, \dots, i_n) \in T_0; \\ \cup_{i_{n+1}} B_{i_1,\dots,i_n,i_{n+1}} & \text{if } (i_1, \dots, i_n) \notin T_0 \text{ and } n \text{ is even}; \\ \cap_{i_{n+1}} B_{i_1,\dots,i_n,i_{n+1}} & \text{if } (i_1, \dots, i_n) \notin T_0 \text{ and } n \text{ is odd}; \end{cases}$$

here i_{n+1} runs over numbers satisfying $(i_1, \dots, i_n, i_{n+1}) \in T$. (Can you prove existence and uniqueness?[1]) Evaluating this function at the root (the empty sequence) we get, by definition, the set B encoded by the given Borel code.

A set $B \subset X$ belongs to the σ-algebra generated by a given set G of subsets of X, if and only if B is encoded by some Borel code (at least one). (Can you prove it?[2])

If the set G (of generators) is of cardinality continuum (or less), then the set of all Borel codes is of cardinality continuum, and therefore the generated σ-algebra is of cardinality continuum (or less).

3b1 Exercise. The following three sets generate the same σ-algebra on $\mathbb{R}$:
$G_1 = \{(-\infty, t] : t \in \mathbb{R}\}$;
G_2: all intervals $I \subset \mathbb{R}$;
G_3: all open sets $U \subset \mathbb{R}$.
Prove it.

3b2 Definition. The *Borel σ-algebra* (on $\mathbb{R}$) is the σ-algebra generated by any of the three sets of 3b1;

a *Borel set* (in $\mathbb{R}$) is a set that belongs to the Borel σ-algebra.

We see that the Borel σ-algebra is of cardinality continuum. Thus, some sets (moreover, most sets) are non-Borel. An example of a non-Borel set may be obtained as follows. One constructs a one-to-one map from Borel codes to reals, and includes to A the real number corresponding to a Borel code if and only if this number does not belong to the encoded Borel set.[3] This

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[1] Hint: assuming the contrary, find an infinite branch.

[2] Hint: if $A_1, A_2, \dots$ are encoded, then $A_1 \cup A_2 \cup \dots$ is encoded.

[3] A kind of Cantor's diagonal argument.

example is explicit but cumbersome. Here is a more elegant example:[1] all real numbers of the form

$$\cfrac{1}{k_1 + \cfrac{1}{k_2 + \dots}}$$

such that some infinite subsequence $(k_{i_1}, k_{i_2}, \dots)$ of the sequence $(k_1, k_2, \dots)$ satisfies the condition: each element is a divisor of the next element.

When a set is Borel, usually, this is easy to prove. When a set is non-Borel, usually, this is hard to prove.[2]

We have two[3] σ-algebras on $\mathbb{R}$: the Lebesgue σ-algebra $\mathcal{L}[\mathbb{R}]$ and the Borel σ-algebra $\mathcal{B}[\mathbb{R}]$. Note that $\mathcal{B}[\mathbb{R}] \subset \mathcal{L}[\mathbb{R}]$, since $\mathcal{L}[\mathbb{R}]$ is some σ-algebra that contains all intervals, and $\mathcal{B}[\mathbb{R}]$ is the least such σ-algebra. Moreover, $\mathcal{B}[\mathbb{R}] \subsetneqq \mathcal{L}[\mathbb{R}]$, for the following reason. The Cantor set

$$C = \Big\{ \sum_{k=1}^\infty 3^{-k} c_k : c_1, c_2, \dots \in \{0, 2\} \Big\}$$

is a compact null set of cardinality continuum. It has more than continuum of subsets; they all are null sets (and moreover, Jordan sets of zero Jordan measure); but only a minority (continuum) of them are Borel sets.

3c Measurability in general; Borel functions

3c1 Definition. Given two measurable spaces (X_1, S_1) and (X_2, S_2), a map $f : X_1 \to X_2$ is called *measurable,* if

$$\forall B \in S_2 \;\; f^{-1}(B) \in S_1 \, .$$

3c2 Lemma. Let S_2 be generated by G_2; then f is measurable if and only if

$$\forall B \in G_2 \;\; f^{-1}(B) \in S_1 \, .$$

Proof. "Only if": trivial. "If": all sets $B \in S_2$ such that $f^{-1}(B) \in S_1$ are a σ-algebra (think, why) that contains G_2; therefore it contains S_2. □

[1] But harder to prove. Due to Lusin, 1927. For detail, see 6c15 in my advanced course "Measurability and continuity". The set is non-Borel but analytic, that is, a continuous image of some Borel set.

[2] In most cases, one proves it by comparing the given set with a known non-Borel set.

[3] There are other notable σ-algebras on $\mathbb{R}$; for example, universally measurable sets. All analytic sets are universally measurable.

You see, the definition "via sets" is quite useful! If you still want to prove the same via Borel codes, try this: if $(T, \{B_i\}_{i \in T_0})$ is a Borel code for B, then $(T, \{f^{-1}(B_i)\}_{i \in T_0})$ is a Borel code for $f^{-1}(B)$.

Combining 3b1 with 3c2 we see that each of the three items (a), (b), (c) of Def. 3a1 is equivalent to 3c1 where $X_2 = \mathbb{R}$ and $S_2 = \mathcal{B}[\mathbb{R}]$.

3c3 Exercise. Do the same for 3a2 (thus introducing $\mathcal{B}[\mathbb{R}^n]$ similarly to 3b2).

In order to prove that a map $f : X \to \mathbb{R}^n$ is measurable if and only if its coordinate functions $f_1, \dots, f_n$ are measurable, combine 3c2 with the following.

3c4 Exercise. $\mathcal{B}[\mathbb{R}^n]$ is generated by sets of the form $\mathbb{R}^{k-1} \times B \times \mathbb{R}^{n-k}$ for $B \in \mathcal{B}[\mathbb{R}]$ and $k = 1, \dots, n$.

Prove it.[1]

3c5 Definition. A Borel function on $\mathbb{R}^d$ is a measurable function on $(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d])$.

3c6 Exercise. (a) If (X_1, S_1), (X_2, S_2), (X_3, S_3) are measurable spaces and $f : X_1 \to X_2$, $g : X_2 \to X_3$ measurable maps, then $g \circ f : X_1 \to X_3$ is a measurable map.

(b) If (X, S) is a measurable space, $f : X \to \mathbb{R}$ a measurable function, and $\varphi : \mathbb{R} \to \mathbb{R}$ a Borel function, then $\varphi \circ f : X \to \mathbb{R}$ is a measurable function.

(c) If (X, S) is a measurable space, $f_1, \dots, f_n : X \to \mathbb{R}$ are measurable functions, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a Borel function, then $\varphi(f_1, \dots, f_n) : x \mapsto \varphi(f_1(x), \dots, f_n(x))$ is a measurable function $X \to \mathbb{R}$.
Prove it.

3c7 Exercise. Every continuous function $\mathbb{R}^d \to \mathbb{R}$ is Borel.

Prove it.[2]

3c8 Corollary. Measurable functions on a measurable space are an algebra.

That is, cf, $f + g$ and fg are measurable whenever f and g are, and $c \in \mathbb{R}$. (Pointwise operations are meant.)

Consider an arbitrary set $A \subset \mathbb{R}^d$. Recall that a relatively open subset of A is a set of the form $A \cap U$ where $U \subset \mathbb{R}^d$ is open. Relatively open sets are a topology on A. Likewise, we define a relatively Borel subset of A as a set of the form $A \cap B$ where $B \subset \mathbb{R}^d$ is Borel.

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[1]Hint: recall 3a2(a) or (b).

[2]Hint: recall 3a2(c).

3c9 Exercise. Relatively Borel sets are a σ-algebra on A, generated by relatively open sets.

Prove it.[1]

Borel functions on A are defined accordingly. Also, 3c6(c) generalizes readily to $X \xrightarrow{f} A \xrightarrow{\varphi} \mathbb{R}$. An example: if $f, g : X \to \mathbb{R}$ are measurable and $\forall x \;\; f(x) > 0$, then $x \mapsto f(x)^{g(x)}$ is measurable.

Often it is convenient to consider functions $X \to [-\infty, +\infty]$. The Borel σ-algebra $\mathcal{B}[-\infty, +\infty]$ on $[-\infty, +\infty]$ is defined evidently.[2]

3c10 Lemma. Let (X, S) be a measurable space. If functions $f_1, f_2, \dots : X \to [-\infty, +\infty]$ are measurable, then their pointwise supremum is measurable.

Proof *(sketch).* $\{x : \sup_k f_k(x) \le t\} = \cap_k \{x : f_k(x) \le t\}$. □

The same holds for $\inf_k f_k$, of course.

3c11 Lemma. Let (X, S) be a measurable space. If functions $f_1, f_2, \dots : X \to [-\infty, +\infty]$ are measurable, then $\limsup_k f_k$ is measurable.

Proof. $\limsup_k f_k = \inf_n \sup_{k>n} f_k$. □

3c12 Corollary. If a sequence of measurable functions converges pointwise, then its limit is a measurable function.

All that holds, in particular, for Borel functions.

Ultimately, every Borel function can be obtained from continuous functions (or step functions, etc.) by taking pointwise limits, and moreover, monotone limits; but the number of zigzags needed is generally unbounded in the same way as in Borel codes of Borel sets.

3d Pushforward measure; distribution

3d1 Exercise. Let (X_1, S_1, μ_1) be a measure space, (X_2, S_2) a measurable space, and $f : X_1 \to X_2$ a measurable map. Then the following function μ_2 on S_2 is a measure:

$$\mu_2(B) = \mu_1\big(f^{-1}(B)\big) \quad \text{for } B \in S_2 \,.$$

Prove it.

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[1] Hint: apply 3c2 to $X_1 = A$, S_1 the σ-algebra generated by relatively open sets, $X_2 = \mathbb{R}^d$, S_2 the Borel σ-algebra, and $f = \mathrm{id} : A \to \mathbb{R}^d$. (Or alternatively, use Borel codes.)

[2] The measurable space $([-\infty, +\infty], \mathcal{B}[-\infty, +\infty])$ is isomorphic to $([-1, 1], \mathcal{B}[-1, 1])$; for instance, the mapping $x \mapsto \tan \frac{\pi}{2} x$ and its inverse $y \mapsto \frac{2}{\pi} \arctan y$ extend as needed, and are Borel.

This μ_2 is called the pushforward measure,

$$\mu_2 = f_* \mu_1 .$$

3d2 Example. If the set $f(X_1) \subset X_2$ is (finite or) countable, then

$$\mu_2(B) = \sum_{k: y_k \in B} p_k ,$$

where $(y_k)_k$ is an enumeration of $f(X_1)$, and $p_k = \mu_2\big(\{y_k\}\big) = \mu_1(f^{-1}(y_k))$.[1] The points y_k are called atoms of μ_2, and p_k their masses. In particular, if f is constant, then μ_2 is a single atom,

$$\mu_2(B) = \begin{cases} p_1 & \text{if } y_1 \in B, \\ 0 & \text{otherwise.} \end{cases}$$

A *probability space*[2] is, by definition, a measure space (X, S, μ) satisfying $\mu(X) = 1$; such μ is called a probability measure. An example: $[0, 1]$ with Lebesgue measure.

Let $\Omega \subset \mathbb{R}^d$ be a measurable set such that $0 < m(\Omega) < \infty$. Then the formula

$$P(A) = \frac{m(A)}{m(\Omega)} \quad \text{for measurable } A \subset \Omega$$

introduces a probability measure P on Ω, often called the uniform distribution on Ω. On the other hand, for a *finite* set Ω the uniform distribution on Ω is $P(A) = \frac{\#A}{\#\Omega}$ (even if $\Omega \subset \mathbb{R}^d$ and points of Ω are distributed in $\mathbb{R}^d$ very unevenly).

Measurable functions on a given probability space $(\Omega, \mathcal{F}, P)$ are called *random variables.*[3] For a random variable $X : \Omega \to \mathbb{R}$ the pushforward measure $X_* P$ is called the (probability) distribution of X, and often denoted by P_X. For n random variables[4] $X_1, \dots, X_n : \Omega \to \mathbb{R}$ their joint distribution is, by definition, the pushforward measure $X_* P$ on $(\mathbb{R}^n, \mathcal{B}[\mathbb{R}^n])$ where $X(\omega) = \big(X_1(\omega), \dots, X_n(\omega)\big)$.

3d3 Example. On the probability space $\Omega = [0, 1)$ (with Lebesgue measure), the random variable $X : \omega \mapsto -\log(1 - \omega)$ has the so-called exponential distribution: $P_X\big((x, \infty)\big) = \mathrm{e}^{-x}$ for $0 \le x < \infty$.

[1] Assuming that these $\{y_k\}$ are measurable.

[2] However, I dislike this standard terminology. I like only so-called standard probability spaces.

[3] "Like the alligator pear that is neither an alligator nor a pear and the biologist's white ant that is neither white nor an ant, the probabilist's random variable is neither random nor a variable." S. Goldberg "Probability: an introduction", Dower 1986, p. 160. (Alligator pear = avocado; white ant = termite.)

[4] By default, all random variables are defined on the same probability space.

3d4 Example. On the same probability space, binary digits $\beta_1, \beta_2, \dots : \Omega \to \{0,1\}$ may be defined by $\sum_{k=1}^{\infty} 2^{-k}\beta_k(\omega) = \omega$, $\liminf_k \beta_k(\omega) = 0$. Treated as random variables, they are used as a model for tossing a fair coin endlessly. For every n, the joint distribution of $\beta_1, \dots, \beta_n$ is the uniform distribution on the finite set $\{0,1\}^n \subset \mathbb{R}^n$.

3d5 Example. Continuing 3d4, consider the random variable

$$X : \omega \mapsto \sum_{k=1}^{\infty} 3^{-k} \cdot 2\beta_k(\omega) \,.$$

Its distribution P_X is nonatomic: $\forall x \in \mathbb{R} \;\; P_X(\{x\}) = 0$, since the map X is one-to-one. Also, $P_X(C) = 1$, where C is the Cantor set; indeed, $X(\Omega) \subset C$. This is an example of a singular distribution: nonatomic but concentrated on a null set (w.r.t. Lebesgue measure).

3d6 Remark. *Every* function $f : [0,1) \to \mathbb{R}$ (measurable or not) is of the form $f = \varphi \circ X$ where $\varphi : \mathbb{R} \to \mathbb{R}$ is measurable, and X is as in 3d5. Indeed, the function

$$\varphi(x) = \begin{cases} f\big(X^{-1}(x)\big) & \text{for } x \in X([0,1)), \\ 0 & \text{otherwise} \end{cases}$$

is measurable, since it vanishes almost everywhere.

This is why in 3c6 we require φ to be Borel.

Note also that *every* subset of $[0,1)$ is of the form $X^{-1}(A)$ for some measurable $A \subset C \subset \mathbb{R}$. This is why a measurable function is a measurable map to $\big(\mathbb{R}, \mathcal{B}[\mathbb{R}]\big)$ rather than $\big(\mathbb{R}, \mathcal{L}[\mathbb{R}]\big)$, and its pushforward measure is defined on $\mathcal{B}[\mathbb{R}]$ rather than $\mathcal{L}[\mathbb{R}]$.

3e Completion

3e1 Definition. Let (X, S, μ) be a measure space.

(a) A *null set* is a set $Z \in S$ such that $\mu(Z) = 0$.

(b) A *sub-null set* is a set contained in some (at least one) null set.

(c) The measure space is *complete,* if every sub-null set is a null set.

For $\mathbb{R}^d$ with Lebesgue measure, null sets are already defined in 2b8 by $m^*(Z) = 0$. Clearly, this does not conflict with 3e1(a); and in this case a set is sub-null if and only if it is null. This measure space is complete.

Now consider the measure space $\big(\mathbb{R}, \mathcal{B}[\mathbb{R}], m|_{\mathcal{B}[\mathbb{R}]}\big)$; here, the Cantor set is null, all its subsets are sub-null, but only a minority (continuum) of them are Borel. This measure space is incomplete.

Given a measure space (X, S, μ) and a set $B \subset X$, we'll say that B is *sandwiched,* if there exist $A, C \in S$ such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$.

3e2 Exercise. A measure space (X, S, μ) is complete if and only if S contains all sandwiched sets.

Prove it.

3e3 Exercise. Let (X, S, μ) be a measure space. Then all sandwiched sets are a σ-algebra $\overline{S} \supset S$, and there exists one and only one measure $\overline{\mu}$ on $\overline{S}$ that extends μ (that is, $\overline{\mu}|_S = \mu$).

Prove it.

The measure space $(X, \overline{S}, \overline{\mu})$ of 3e3 is complete (think, why). It is called the *completion* of (X, S, μ). It is equal to (X, S, μ) if and only if (X, S, μ) is complete.

The completion of $\big(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d], m|_{\mathcal{B}[\mathbb{R}^d]}\big)$ is the complete space $\big(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m\big)$, since every measurable set of finite measure is sandwiched between two Borel sets (a countable union of compact sets, and a countable intersection of open sets, recall 2c3), and every measurable set is a countable union of measurable sets of finite measure. See also 2d8(b).[1]

By default, "a measure on $\mathbb{R}^d$" means either a Borel measure, that is, measure on $\mathcal{B}[\mathbb{R}^d]$, or its completion. Note that the completed σ-algebra $\overline{S}_\mu$ depends heavily on the measure μ. If μ is purely atomic (as in 3d2), then $\overline{S}_\mu$ contains *all* subsets of $\mathbb{R}^d$. If $\mu = m$ (or $m|_{\mathcal{B}[\mathbb{R}]}$), then $\overline{S}_\mu$ contains (in particular) all subsets of the Cantor set, which is not the case for the singular measure of 3d5.

3f Finite, locally finite, σ-finite etc.

3f1 Definition. A measure μ on a measurable space (X, S) is called

(a) finite, if $\mu(X) < \infty$;

(b) σ-finite, if there exist $A_1, A_2, \dots \in S$ such that $\forall k \;\; \mu(A_k) < \infty$ and $\cup_k A_k = X$.

These properties are invariant under completion (think, why).

The Lebesgue measure on $\mathbb{R}^d$ is σ-finite, and not finite.

A measure μ on $\mathbb{R}^d$ is called *locally finite,* if $\mu(B) < \infty$ for all bounded Borel sets B. (Invariant under completion, still.) A locally finite measure is σ-finite, but the converse is generally wrong.

3f2 Example. Let $\mu(A) = \#(A \cap \mathbb{Q})$ be the number of rational numbers in A. Then μ is a measure on $\mathbb{R}$, σ-finite but not locally finite (think, why). It is of the form $f_* m$; just take the step function, $f(x) = r_k$ for $x \in [k, k+1)$, where $(r_k)_{k \in \mathbb{Z}}$ is an enumeration of $\mathbb{Q}$.

[1] A Jordan set need not be Borel, but is sandwiched between its interior and closure.

3f3 Exercise. Given $\alpha > 0$, consider the function $f : \mathbb{R} \to \mathbb{R}$ linear on every $[k, k+1]$ and satisfying $f(k) = (-1)^k |k|^\alpha$ for $k \in \mathbb{Z}$.

(a) If $\alpha > 1$, then the measure $\mu = f_* m$ is locally finite (and not finite);

(b) if $\alpha \le 1$, then

$$\mu(B) = \begin{cases} 0 & \text{if } m(B) = 0, \\ \infty & \text{otherwise} \end{cases}$$

for all Borel B; this μ is not σ-finite.

Prove it.

Even if $f_* m$ is not σ-finite, it is still the sum $\mu_1 + \mu_2 + \dots$ of some finite measures μ_k (think, why). Even this weak property is violated by the counting measure $A \mapsto \#(A)$. Indeed, every finite measure under the counting measure is concentrated on a countable set; but the counting measure is not.

And the worst case is, of course,

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

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