Fine Structure of EPR State and Universal Quantum Correlation

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The problem

If a pair of spin-1/2 particles in the singlet state

$$\Psi = \tfrac{1}{\sqrt{2}}|+\rangle|-\rangle - \tfrac{1}{\sqrt{2}}|-\rangle|+\rangle \tag{1}$$

is given, then the equality

$$\langle A_1 A_2\rangle + \langle A_1 B_2\rangle + \langle B_1 A_2\rangle - \langle B_1 B_2\rangle = 2\sqrt{2} \tag{2}$$

takes place for some spin projections A_1, B_1 of the first particle and A_2, B_2 of the second (taking on the two values ± 1). This is the maximal violation of Bell-CHSH inequality within the quantum theory.

If a pair of spinless particles in EPR state

$$\psi(x_1, x_2) = \sqrt{\delta(x_1 - x_2)} \tag{3}$$

is given, does the equality (2) hold for some two-valued observables A_1, B_1 for the first particle and A_2, B_2 for the second? Yes, it does (Summers and Werner [1]).

A non-singlet entangled spin state

$$\Psi = \alpha|+\rangle|-\rangle + \beta|-\rangle|+\rangle\,, \qquad |\alpha| \ne |\beta|\,, \tag{4}$$

was used by Hardy [2] for the following spectacular observation: A_1, B_1, A_2, B_2 can be chosen so that each of the three inequalities

$$A_1 \le A_2\,, \quad A_2 \le B_1\,, \quad B_1 \le B_2 \tag{5}$$

holds with probability 1, and nevertheless the inequality

$$A_1 \le B_2 \tag{6}$$

is violated with a positive probability. In other words,

$$\begin{aligned} &\langle (1+A_1)(1-A_2)\rangle = 0\,,\\ &\langle (1-B_1)(1+A_2)\rangle = 0\,,\\ &\langle (1+B_1)(1-B_2)\rangle = 0\,,\\ \\ &\langle (1+A_1)(1-B_2)\rangle > 0\,. \end{aligned} \tag{7}$$

Is this situation (7) possible for the EPR state (3)? A positive answer follows immediately from a general result announced in my work [3] and proved here.

First of all, the main idea will be presented informally, with no attention to mathematical rigor when dealing with delta-functions (as in (3)).

The main idea of a solution

Representing the coordinate x of a spinless particle via its integral part $[x]$ and fractional part $\{x\}$,

$$x = [x] + \{x\} , \tag{8}$$

we may write

$$\sqrt{\delta(x_1 - x_2)} = \sqrt{\delta([x_1] - [x_2])} \cdot \sqrt{\delta(\{x_1\} - \{x_2\})} . \tag{9}$$

Of course, the expression $\delta([x_1] - [x_2])$ contains the discrete delta, taking the values 0 and 1 (thus, the square root may be dropped this time), while $\delta(\{x_1\} - \{x_2\})$ contains Dirac's delta function.

Further, the integral part $[x]$ may be represented via its even part $2[x/2]$ and the remainder (0 or 1); the latter is the residual $[x]_2 = [x] \bmod 2$:

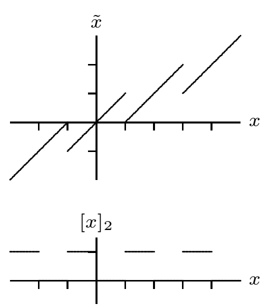
$$[x] = 2\left[\tfrac{x}{2}\right] + [x]_2 . \tag{10}$$

Introduce

$$\tilde{x} = \left[\tfrac{x}{2}\right] + \{x\} , \tag{11}$$

and observe a one-one correspondence between x and the pair $(\tilde{x}, [x]_2)$:

$$x = 2[\tilde{x}] + \{\tilde{x}\} + [x]_2 . \tag{12}$$

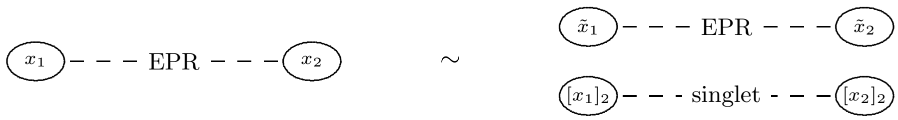


These $\tilde{x}$ and $[x]_2$ may be treated as two degrees of freedom, one being continuous, the other discrete.

The trivial equality $\delta([x_1] - [x_2]) = \delta([x_1/2] - [x_2/2]) \cdot \delta([x_1]_2 - [x_2]_2)$, combined with (9), gives

$$\sqrt{\delta(x_1 - x_2)} = \sqrt{\delta(\tilde{x}_1 - \tilde{x}_2)} \cdot \sqrt{\delta([x_1]_2 - [x_2]_2)} . \tag{13}$$

This means that the new degrees of freedom are uncorrelated:

(14) 

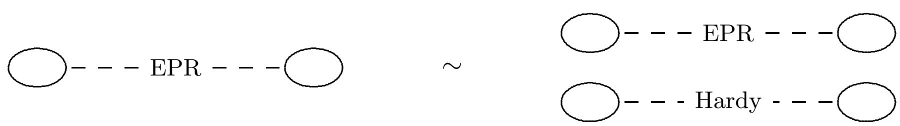
An EPR pair splits into a singlet pair and another EPR pair!

The first consequence is the above-mentioned result of Summers and Werner: the quantum bound (2) can be reached by EPR state. In other words, the quantum correlation matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{15}$$

can be implemented by EPR state. The second consequence: *any* quantum correlation matrix (of any size!) can be implemented by EPR state; see [3], Sect. 3.

However, Hardy's case (7) involves not only correlations $\langle A_1 A_2 \rangle$, $\langle A_1 B_2 \rangle$, ... but also linear terms $\langle A_1 \rangle$, ... This is why it is not covered by the above universality property of the EPR state. Can we split an EPR pair into a non-singlet entangled pair (4) and another EPR pair?

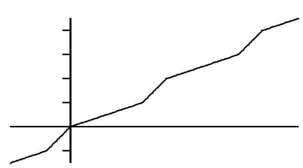
(16) 

This can be done, in the same way as (14); the point is that the following state is isomorphic to EPR state:

$$\psi(y_1, y_2) = \sqrt{\delta(y_1 - y_2)} \cdot (\alpha + \beta [y]_2) \, . \tag{17}$$

Here α, β are arbitrary positive constants, and $[y]_2$ is either $[y_1]_2$, or $[y_2]_2$, which is the same due to $\delta(y_1 - y_2)$. This "piecewise EPR" state (17) can be obtained from EPR state (3) by a piecewise linear transformation of coordinates:

$$\begin{gathered} y_1 = f(x_1) \, , \qquad y_2 = f(x_2) \, , \\ \delta(y_1 - y_2) = \frac{\delta(x_1 - x_2)}{f'(x)} \, . \end{gathered} \tag{18}$$

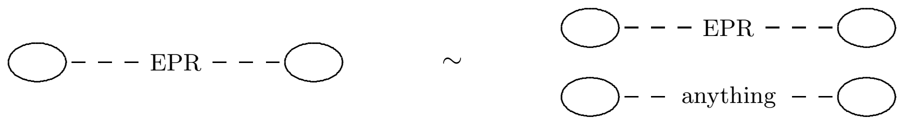


Thus, using (13), $\sqrt{\delta(x_1 - x_2)} = \sqrt{\delta(y_1 - y_2)} \cdot (\alpha + \beta [y]_2) = \sqrt{\delta(\tilde{y}_1 - \tilde{y}_2)} \cdot \sqrt{\delta([y_1]_2 - [y_2]_2)} \cdot (\alpha + \beta [y]_2)$, which means (16).

Using more than two pieces, we can replace (4) with

$$\Psi = \alpha_1|1\rangle|1\rangle + \alpha_2|2\rangle|2\rangle + \dots \tag{19}$$

which is the general form of a state vector of a two-component quantum system, well-known as Schmidt decomposition. Thus:

(20) 

EPR state is universal among all two-body quantum states!

Some subtleties

Of course, $\sqrt{\delta(x_1 - x_2)}$ is not an element of $L_2(\mathbb{R}^2)$; some approximation is needed. Usually, any sequence of state vectors $\Psi_n \in L_2(\mathbb{R}^2)$, satisfying the following condition, is treated as "asymptotically EPR":

$$\langle \Psi_n|(Q_1 - Q_2)^2|\Psi_n\rangle \to 0 \quad \text{and} \quad \langle \Psi_n|(P_1 + P_2)^2|\Psi_n\rangle \to 0 \tag{21}$$

for $n \to \infty$; here Q_1, Q_2 are coordinate operators, P_1, P_2 are momentum operators. Uncertainty relations

$$\Delta_n(Q_1 - Q_2) \cdot \Delta_n(P_1 - P_2) \ge h\,, \quad \Delta_n(Q_1 + Q_2) \cdot \Delta_n(P_1 + P_2) \ge h$$

(Δ_n being the uncertainty for Ψ_n) imply $\langle \Psi_n|(Q_1+Q_2)^2|\Psi_n\rangle \to \infty$, $\langle \Psi_n|(P_1-P_2)^2|\Psi_n\rangle \to \infty$. The products

$$S_n = \frac{1}{h^2}\Delta_n(Q_1 - Q_2) \cdot \Delta_n(P_1 - P_2) \cdot \Delta_n(Q_1 + Q_2) \cdot \Delta_n(P_1 + P_2) \tag{22}$$

may be bounded or unbounded, when $n \to \infty$. The minimal value $S_n = 1$ corresponds to a coherent state.

Interestingly, coherent states are not fit for the present work. The formal relation

$$a\delta(ax) = \delta(x) \tag{23}$$

was used in (18). Its formal consequence

$$\int \sqrt{a\delta(ax)}\sqrt{\delta(x)}\,dx = 1 \tag{24}$$

is important. Let $f_n \to \delta$ in the sense that

$$\int f_n(x)\varphi(x)\,dx \to \varphi(0) \tag{25}$$

for any smooth test function φ. Does it mean that

$$\int \sqrt{af_n(ax)}\sqrt{f_n(x)}\,dx \to 1 \quad ? \tag{26}$$

In no way! Usually this is not the case. The relation (26) requires that f_n are more or less similar to the following:

$$f_n(x) = \begin{cases} \dfrac{1}{2\ln n} \cdot \dfrac{1}{x} & \text{when } \frac{1}{n} < |x| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

We see that there is a "fine structure" behind the notion of "EPR state," and this may be of value for some quantum correlations.

The entangled wave function $\sqrt{f_n(x_1 - x_2)}$ with f_n as in (27) has its Schmidt decomposition; the set of its coefficients is asymptotically dense for large n. Maybe, this fact is responsible for the universality property. I do not know, whether this universality is compatible with boundedness of S_n (see (22)), or not.

References

[1] S.J. Summers, R. Werner (1987) Bell's inequalities and quantum field theory. II. Bell's inequalities are maximally violated in the vacuum. *J. Math. Phys.* **28**:10, 2448–2456.

[2] L. Hardy (1992) A quantum optical experiment to test local realism. *Phys. Letters A* **167**, 17–23.

[3] B.S. Tsirelson (1993) Some results and problems on quantum Bell-type inequalities. *Hadronic Journal Suppl.* **8**, 329–345.