THEORY PROBAB. APPL.
Vol. 31, No. 3

Translated from Russian Journal

A GEOMETRIC APPROACH TO MAXIMUM LIKELIHOOD ESTIMATION FOR INFINITE-DIMENSIONAL LOCATION. III*

B. S. TSIREL'SON

(*Translated by Laila Ellis*)

In the second part [2] of our series of articles we formulated four theorems without proofs. Their proofs are given here, in this third part, along with the necessary lemmas; the theorems are not stated here, nor are the definitions. We continue to use the objects and notation introduced in the first part [1]: E, γ, E_0, $\langle\theta, x\rangle$, $\|\theta\|$, γ_θ, $\gamma_{\theta,\sigma}$, $\mathscr{L}_\sigma(\theta, x)$, V, $C_1(V)$, $C_2(V, \theta)$, as well as $B(\theta, r)$, and those introduced in [2]: $\mathcal{M}_k(V)$ and γ_V, and those appearing in the statements of Theorems 1–4: F_k, $K(\theta)$ and the ρ-correlated random elements. As in [2], the set V is assumed to be convex and such that $C_1(V)<+\infty$, and consequently $C_1(V)=0$. Recall that

$$\mathcal{M}_k(V)\leqq \mathcal{M}_1^k(V), \tag{1}$$

as well as

$$\mathcal{M}_1(V\cap B(\theta, r))\leqq r\mathcal{M}_1(V\cap B(\theta, 1)) \quad \text{for } r\geqq 1, \quad \theta\in V; \tag{2}$$

see [2, formulas (4) and (1)]. A significant role will be played by the following function on E:

$$\varphi(x)=\sup_{\theta\in V}\log \mathscr{L}(\theta, x)=\sup_{\theta\in V}(\langle\theta, x\rangle-\tfrac{1}{2}\|\theta\|^2);$$

as Theorem 1 in [1] shows, the function φ is well defined and finite γ_σ-a.e. for any σ. According to formula (5) in [2],

$$\frac{\gamma_V(dx)}{\gamma(dx)}=\exp\varphi(x). \tag{3}$$

Finally, by definition, $a_+=\max(a, 0)$; accordingly, $(a-b)_+^2$ is equal to $(a-b)^2$ for $a\geqq b$ and to 0 for $a\leqq b$.

Let us begin with the finite-dimensional case. We assume that $E=\mathbf{R}^n$, $\gamma=\gamma^n$, the metric is Euclidean, and $\hat\theta(x)$ is the closest point in V to x. Let us point out some useful inequalities:

$$\langle\hat\theta(x)-\theta, x-\hat\theta(x)\rangle\geqq 0 \tag{4}$$

for any $x\in\mathbf{R}^n$, $\theta\in V$;

$$\langle\hat\theta(x)-\hat\theta(y), x-y\rangle\geqq\|\hat\theta(x)-\hat\theta(y)\|^2 \tag{5}$$

for any $x, y\in\mathbf{R}^n$; and if $0\in V$, then

$$\langle\hat\theta(x), x\rangle\geqq\|\hat\theta(x)\|^2 \tag{6}$$

for any $x\in\mathbf{R}^n$. Inequality (4) is geometrically obvious: the hyperplane passing through $\hat\theta(x)$ perpendicular to the segment joining $\hat\theta(x)$ with x separates V from x.

* Received by the editors June 18, 1984.

Inequality (6) is obtained from (4) for $\theta=0$. Inequality (5) is derived as follows: for $\theta=\hat{\theta}(y)$ we find from (4) that $\langle\hat{\theta}(x)-\hat{\theta}(y), x-\hat{\theta}(x)\rangle\geqq 0$; we switch the places of x and y, add the resulting inequality to the original one

$$\langle\hat{\theta}(y)-\hat{\theta}(x), y-\hat{\theta}(y)\rangle+\langle\hat{\theta}(x)-\hat{\theta}(y), x-\hat{\theta}(x)\rangle\geqq 0;$$

$$\langle\hat{\theta}(x)-\hat{\theta}(y), x-\hat{\theta}(x)-y+\hat{\theta}(y)\rangle\geqq 0;$$

and obtain (5).

Remark 1. Inequalities (4), (5) and (6) remain valid in the infinite-dimensional case (for almost all x and y) if $V\in GC$.

We shall study the function φ in the finite-dimensional case. We have

(7)
$$\varphi(x)=\tfrac{1}{2}\sup_{\theta\in V}(\|x\|^2-\|x-\theta\|^2)$$
$$=\tfrac{1}{2}\|x\|^2-\tfrac{1}{2}\|x-\hat{\theta}(x)\|^2.$$

This function is convex on $\mathbf{R}^n$ and continuously differentiable; it is not hard to calculate its differential:

(8)
$$D\varphi(x,h)=\langle\hat{\theta}(x), h\rangle.$$

Now let us suppose that V is a convex solid with a C^2-smooth boundary ∂V. Then the principal curvatures $\kappa_1(\theta),\cdots,\kappa_{n-1}(\theta)$ of the surface ∂V are defined at each point $\theta\in\partial V$; their order of enumeration is of no consequence for us. In this case also φ will be C^2-smooth: $\varphi(x+h)=\varphi(x)+D\varphi(x,h)+\frac{1}{2}D^2\varphi(x,h)+o(\|h\|^2)$; $D^2\varphi(x,h)$ is a quadratic form (in h); let us consider its characteristic values $\lambda_1(x),\cdots,\lambda_n(x)$. For $x\in\mathbf{R}^n\setminus V$ one of them is equal to zero since the form is singular in the direction of the vector $x-\hat{\theta}(x)$; we shall assume that $\lambda_n(x)=0$. The other characteristic values are associated with the principal curvatures:

(9)
$$\lambda_i(x)(1+\|x-\hat{\theta}(x)\|\kappa_i(\hat{\theta}(x)))=1.$$

Clearly, $0<\lambda_i(x)\leqq 1$. Of course for $x\in V\setminus\partial V$ the expression $\kappa_i(\hat{\theta}(x))$ becomes meaningless; in this case, $\lambda_i(x)=1$ for $i=1,\cdots,n$. It is clear from (8) that the $\lambda_i(x)$ are also the eigenvalues of the differential of the mapping $\hat{\theta}$ at x. The following lemma links these numbers to the thicknesses.

LEMMA 1. *For any* $z\in(-\infty,+\infty)$,

$$\int_{\mathbf{R}^n}\prod_{i=1}^{n}(1+z\lambda_i(x))\gamma_V(dx)=\sum_{k=0}^{n}\frac{1}{k!}\mathcal{M}_k(V)(1+z)^k.$$

Proof. The one-to-one mapping $x\to(\hat{\theta}(x),\|x-\hat{\theta}(x)\|)$ of the set $\mathbf{R}^n\setminus V$ onto $\partial V\times(0,+\infty)$ has Jacobian $\lambda_1(x)\cdots\lambda_{n-1}(x)$. Hence taking (3) and (9) into account we obtain the following change-of-variables formula (Ψ is any function on $\partial V\times(0,+\infty)$, for which the indicated integrals exist):

$$\int_{\mathbf{R}^n\setminus V}\Psi(\hat{\theta}(x),\|x-\hat{\theta}(x)\|)\gamma_V(dx)$$
$$=(2\pi)^{-n/2}\int_{\partial V}\int_0^\infty\Psi(\theta,t)\exp(-t^2/2)\prod_{i=1}^{n-1}(1+t\kappa_i(\theta))\,dtS(d\theta);$$

and $S(d\theta)$ is the area element on ∂V. Let us apply this formula to the integral given in the hypotheses of the lemma, after first reducing it to the necessary form with the aid of formula (9):

$$\int_{\mathbf{R}^n}\prod_{i=1}^{n}(1+z\lambda_i(x))\gamma_V(dx)=\int_V(1+z)^n\gamma_V(dx)$$

$$+\int_{\mathbf{R}^n\setminus V}\prod_{i=1}^{n-1}\left(1+\frac{z}{1+\|x-\hat\theta(x)\|\kappa_i(\hat\theta(x))}\right)\gamma_V(dx)$$

$$=(1+z)^n(2\pi)^{-n/2}\operatorname{meas}_n V+(2\pi)^{-n/2}\int_{\partial V}\int_0^\infty\prod_{i=1}^{n-1}\left(1+\frac{z}{1+t\kappa_i(\theta)}\right)$$

$$\cdot\exp\left(-\frac{t^2}{2}\right)\prod_{i=1}^{n-1}(1+t\kappa_i(\theta))\,dtS(d\theta).$$

The first term is equal to $(n!)^{-1}\mathcal{M}_n(V)(1+z)^n$ (see [3, 3.5.2]). Using the elementary symmetric functions of the principal curvatures

$$\sigma_\nu(\theta)=\sum_{1\le i_1<\cdots<i_\nu\le n-1}\kappa_{i_1}(\theta)\cdots\kappa_{i_\nu}(\theta),$$

we transform the second term as follows:

$$(2\pi)^{-n/2}\int_{\partial V}\int_0^\infty\prod_{i=1}^{n-1}(1+t\kappa_i(\theta)+z)\exp(-t^2/2)\,dtS(d\theta)$$

$$=(2\pi)^{-n/2}\int_{\partial V}\int_0^\infty\sum_{k=0}^{n-1}(1+z)^k t^{n-1-k}\sigma_{n-1-k}(\theta)\exp(-t^2/2)\,dtS(d\theta)$$

$$=(2\pi)^{-n/2}\sum_{k=0}^{n-1}(1+z)^k\left(\int_{\partial V}\sigma_{n-k-1}(\theta)S(d\theta)\right)$$

$$\cdot\left(\int_0^\infty t^{n-1-k}\exp(-t^2/2)\,dt\right)$$

$$=(2\pi)^{-n/2}\sum_{k=0}^{n-1}(1+z)^k 2^{(n-k-2)/2}\Gamma\left(\frac{n-k}{2}\right)\int_{\partial V}\sigma_{n-k-1}(\theta)S(d\theta).$$

A formula linking the surface integral of σ_{n-k-1} with the kth transverse measure is known (e.g., see [4, formula 21, p. 142]), which in our notation becomes

$$\mathcal{M}_k(V)=2^{-(k+2)/2}\pi^{-n/2}k!\,\Gamma\left(\frac{n-k}{2}\right)\int_{\partial V}\sigma_{n-k-1}(\theta)S(d\theta).$$

We now have

$$\int_{\mathbf{R}^n}\prod_{i=1}^{n}(1+z\lambda_i(x))\gamma_V(dx)$$

$$=\frac{1}{n!}\mathcal{M}_n(V)(1+z)^n$$

$$+(2\pi)^{-n/2}\sum_{k=0}^{n-1}(1+z)^k 2^{(n-k-2)/2}\Gamma\left(\frac{n-k}{2}\right)\frac{2^{(k+2)/2}\pi^{n/2}}{k!\,\Gamma((n-k)/2)}\mathcal{M}_k(V)$$

$$=\sum_{k=0}^{n}\frac{1}{k!}\mathcal{M}_k(V)(1+z)^k.$$

Remark 2. If the surface ∂V is piecewise C^2-smooth, then the second derivatives of φ and their characteristic numbers $\lambda_i(x)$ are piecewise continuous. It is not hard to show by an approximation that Lemma 1 is preserved for this case. Further, if one takes a polyhedron as V, then all the $\lambda_i(x)$ become zeros and ones, and our Lemma 1 yields Lemma 1 in [2].

LEMMA 2. *For any $\theta \in V$, $\sigma > 0$ and $z > -1$,*

$$\int \prod_{i=1}^{n} (1 + z\lambda_i(x))\gamma_{\theta,\sigma}(dx) \leqq \exp\left(\frac{1+z}{\sigma}\mathcal{M}_1(V)\right).$$

Proof. We can assume that $\theta = 0$ and $\sigma = 1$; the general case can be reduced to this one via a shift and a dilation. Applying Lemma 1 and inequality (1) we obtain

$$\begin{aligned}\int \prod_{i=1}^{n} (1 + z\lambda_i(x))\gamma(dx) &\leqq \int \prod_{i=1}^{n} (1 + z\lambda_i(x))\gamma_V(dx) \\ &= \sum_{k=0}^{n} \frac{1}{k!}\mathcal{M}_k(V)(1+z)^k \\ &\leqq \sum_{k=0}^{n} \frac{1}{k!}((1+z)\mathcal{M}_1(V))^k \\ &\leqq \exp((1+z)\mathcal{M}_1(V)),\end{aligned}$$

as required.

The following two lemmas estimate the distance of the maximum likelihood estimate (MLE) from the true value of the parameter. The space E can again be both finite-dimensional and infinite-dimensional.

LEMMA 3. *Let $V \subset E_0$ be a convex GB-compact set, $\theta \in V$ and $\sigma > 0$. Then, for any $a \in [0, 1]$,*

$$\int \exp\left(\frac{a(2-a)}{2\sigma^2}\|\hat{\theta}(x) - \theta\|^2\right)\gamma_{\theta,\sigma}(dx) \leqq \exp\left(\frac{a}{\sigma}\mathcal{M}_1(V)\right).$$

Proof. We can assume that $\theta = 0$ and $\sigma = 1$. The general case can be reduced to this one via a shift and a dilation. Also, we can assume that V is finite-dimensional; for the general case can be reduced to this one by an approximation from the inside in view of Remark 3 in [1]. We apply inequality (6), formula (3) and Corollary 1 in [2]:

$$\begin{aligned}\int \exp\left(\frac{a(2-a)}{2}\|\hat{\theta}(x)\|^2\right)\gamma(dx) &= \int \exp\left(a\|\hat{\theta}(x)\|^2 - \frac{a^2}{2}\|\hat{\theta}(x)\|^2\right)\gamma(dx) \\ &\leqq \int \exp\left(a\langle\hat{\theta}(x), x\rangle - \frac{a^2}{2}\|\hat{\theta}(x)\|^2\right)\gamma(dx) \\ &\leqq \int \exp \sup_{\theta \in aV}\left(\langle\theta, x\rangle - \frac{1}{2}\|\theta\|^2\right)\gamma(dx) \\ &= \int \gamma_{aV}(dx) = \gamma_{aV}(E) \\ &\leqq \exp(\mathcal{M}_1(aV)) = \exp(a\,\mathcal{M}_1(V)).\end{aligned}$$

COROLLARY 1. *Under the hypotheses of Lemma 3,*

$$\gamma_{\theta,\sigma}\{x \in E: \|\hat{\theta}(x) - \theta\| \geqq r\} \leqq e^{-u}$$

for any positive r and u such that

$$\sqrt{2u} = r/\sigma - \mathcal{M}_1(V)/r.$$

Proof. For any $a \in [0, 1]$ we have

$$\gamma_{\theta,\sigma}\{x \in E: \|\hat{\theta}(x) - \theta\| \geqq r\}$$
$$\leqq \exp\left(-\frac{a(2-a)}{2\sigma^2} r^2\right) \int \exp\left(\frac{a(2-a)}{2\sigma^2} \|\hat{\theta}(x) - \theta\|^2\right) \gamma_{\theta,\sigma}(dx)$$
$$\leqq \exp\left(-\frac{a(2-a)}{2\sigma^2} r^2 + \frac{a}{\sigma} \mathcal{M}_1(V)\right).$$

We assume that $\mathcal{M}_1(V) \leqq \sigma r^2$ (otherwise there is nothing to prove). The minimum with respect to a is attained for $a = 1 - \mathcal{M}_1(V)/(\sigma r^2)$; substituting this a, we obtain on the right-hand side

$$\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - \frac{\mathcal{M}_1(V)}{r}\right)^2\right).$$

Remark 3. For nonconvex V the corresponding weaker estimate is derived from different considerations in [1, Thm. 2(b)]; in the notation used here this estimate takes on the form

$$\sqrt{2u} = \frac{r}{R(V, \theta)} - \left(\frac{r}{2\sigma} - \frac{\mathcal{M}_1(V)}{r}\right).$$

LEMMA 4. *Let $V \subset E_0$ be a closed convex set, $\theta \in V$, $\sigma > 0$, $\mathcal{M}_1(V \cap B(\theta, 1)) = C_2 < +\infty$. Then, for all $r \geqq \max(\sigma C_2, \sqrt{\sigma C_2})$,*

$$\gamma_{\theta,\sigma}\{x \in E: \|\hat{\theta}(x) - \theta\| \geqq r\} \leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{\min(r, 1)}\right)^2\right).$$

Proof. First step. Let $0 < r_0 < r_1 < \cdots$, $r_n \to +\infty$, and $r_1 \geqq 1$; applying Corollary 1 to the sets $V_{n+1} = V \cap B(\theta, r_{n+1})$ and noting that $\mathcal{M}_1(V_{n+1}) \leqq r_{n+1} C_2$ according to (2) (it is here that the fact that $r_1 \geqq 1$ is essential), we obtain

$$\gamma_{\theta,\sigma}\{x \in E: \|\hat{\theta}(x) - \theta\| \geqq r_0\} \leqq \sum_{n=0}^{\infty} \gamma_{\theta,\sigma}\{x \in E: r_n \leqq \|\hat{\theta}(x) - \theta\| \leqq r_{n+1}\}$$
$$\leqq \sum_{n=0}^{\infty} \gamma_{\theta,\sigma}\{x \in E: \|\hat{\theta}(x, V_{n+1}) - \theta\| \geqq r_n\}$$
$$\leqq \sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{r_n}{\sigma} - \frac{r_{n+1} C_2}{r_n}\right)_+^2\right).$$

At the second step we shall prove that for any $r_0 \geqq (C_2 + 2)\sigma$ one can choose $r_1, r_2, \cdots$, such that $r_0 < r_1 < r_2 < \cdots$, $r_n \to +\infty$ and

$$\sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{r_n}{\sigma} - \frac{r_{n+1} C_2}{r_n}\right)_+^2\right) \leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - C\right)^2\right).$$

Let us convince ourselves that this implies the assertion of the lemma. For $r \leqq (C_2 + 2)\sigma$ there is nothing to prove, for then

$$9 \exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{\min(r, 1)}\right)_+^2\right) \geqq 9 \exp(-2) > 1.$$

For $r \geqq \max(1, (C_2 + 2)\sigma)$ we put $r_0 = r$ and obtain the desired at once. There remains only the case $(C_2 + 2)\sigma \leqq r \leqq 1$. It will become clear at the second step that $r_1 = f(r_0)$, where f is a continuous function on $[(C_2+2)\sigma, +\infty)$. If $f(r_0) \geqq 1$ for all $r_0 \in [(C_2+2)\sigma, 1]$, then we again put $r_0 = r$ and obtain

$$\gamma_{\theta,\sigma}\{x \in E: \|\hat{\theta}(x) - \theta\| \geqq r\} \leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - C_2\right)^2\right)$$

$$\leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{r}\right)_+^2\right).$$

There remains the case where there exists an $r' \in [(C_2+2)\sigma, 1]$, such that $f(r'') \geqq 1$ for all $r'' \in [r', 1]$, $f(r') = 1$, and $r \in [(C_2+2)\sigma, r']$. In this case we put $r_0 = r'$ and derive from the inequality

$$\sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{r_n}{\sigma} - \frac{r_{n+1}C_2}{r_n}\right)_+^2\right) \leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - C_2\right)^2\right)$$

that

$$\exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - \frac{C_2}{r_0}\right)_+^2\right) + \Sigma_1 \leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - C_2\right)^2\right)$$

$$\leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - \frac{C_2}{r_0}\right)_+^2\right),$$

where

$$\Sigma_1 = \sum_{n=1}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{r_n}{\sigma} - \frac{r_{n+1}C_2}{r_n}\right)_+^2\right);$$

thus,

$$\Sigma_1 \leqq 8 \exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - \frac{C_2}{r_0}\right)_+^2\right).$$

We now have

$$\gamma_{\theta,\sigma}\{x \in E: \|\hat{\theta}(x) - \theta\| \geqq r\} \leqq \exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{r}\right)_+^2\right) + \Sigma_1$$

$$\leqq \exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{r}\right)_+^2\right) + 8 \exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - \frac{C_2}{r_0}\right)_+^2\right)$$

$$\leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{r}\right)_+^2\right),$$

as required.

Second step. Given $r_0 \geqq (C_2+2)\sigma$; it is required to construct $r_1, r_2, \cdots$ such that $r_0 < r_1 < r_2 < \cdots$, $r_n \to +\infty$ and

$$\sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{r_n}{\sigma} - \frac{r_{n+1}C_2}{r_n}\right)_+^2\right) \leqq 9 \exp\left(-\frac{1}{2}\left(\frac{r_0}{\sigma} - C_2\right)^2\right).$$

We make the substitution $r_n=(C_2+t_n)\sigma$; and then we have

$$\frac{r_n}{\sigma}-\frac{r_{n+1}C_2}{r_n}=t_n+C_2-\frac{(t_{n+1}+C_2)C_2}{t_n+C_2}$$

$$=t_n-\frac{C_2}{C_2+t_n}(t_{n+1}-t_n)$$

$$\geqq t_n-(t_{n+1}-t_n)=2t_n-t_{n+1}.$$

Thus, for given $t_0\geqq 2$ it suffices to construct $t_1, t_2, \cdots$, such that $t_0<t_1<t_2<\cdots, t_n\to+\infty$, and

$$\sum_{n=0}^{\infty}\exp\left(-\frac{1}{2}(2t_n-t_{n+1})_+^2\right)\leqq 9\exp\left(-\frac{1}{2}t_0^2\right).$$

Take $p_0\geqq 1$ such that $p_0+1/p_0=t_0$. Further, take $p_n\geqq 1$ such that

$$p_n\exp(-\tfrac{1}{2}p_n^2)=2^{-n}p_0\exp(-\tfrac{1}{2}p_0^2).$$

Then clearly p_n increases to $+\infty$. For $n=1, 2, \cdots$ set

$$t_n=2^nt_0-\sum_{k=0}^{n-1}2^{n-k-1}p_k,$$

We then have $2t_n-t_{n+1}=p_n$. At the third step we shall see that $\sum_{n=0}^{\infty}2^{-n}p_n<2(p_0+1/p_0)$. We have

$$t_{n+1}-t_n=2^{n+1}t_0-\sum_{k=0}^{n}2^{n-k}p_k-2^nt_0+\sum_{k=0}^{n-1}2^{n-k-1}p_k$$

$$=2^nt_0-\sum_{k=0}^{n-1}2^{n-k-1}p_k-p_n\geqq 2^nt_0-\sum_{k=0}^{\infty}2^{n-k-1}p_k$$

$$=2^n\left(t_0-\frac{1}{2}\sum_{k=0}^{\infty}2^{-k}p_k\right),$$

and the last expression in the parentheses is positive. It is now evident that t_n increases to $+\infty$. We have

$$\sum_{n=0}^{\infty}\exp\left(-\frac{1}{2}(2t_n-t_{n+1})_+^2\right)=\sum_{n=0}^{\infty}\exp\left(-\frac{1}{2}p_n^2\right)$$

$$=\sum_{n=0}^{\infty}2^{-n}\frac{p_0}{p_n}\exp\left(-\frac{1}{2}p_0^2\right)\leqq 2\exp\left(-\frac{1}{2}p_0^2\right).$$

But

$$9\exp\left(-\frac{1}{2}t_0^2\right)=9\exp\left(-\frac{1}{2}(p_0^2+2+p_0^{-2})\right)\geqq 9\exp\left(-\frac{1}{2}(p_0^2+3)\right)$$

$$=9\exp\left(-\frac{3}{2}\right)\exp\left(-\frac{1}{2}p_0^2\right)>2\exp\left(-\frac{1}{2}p_0^2\right)$$

$$\geqq\sum_{n=0}^{\infty}\exp\left(-\frac{1}{2}(2t_n-t_{n+1})_+^2\right).$$

Third step. Let us show that $\sum_{n=0}^{\infty} 2^{-n} p_n < 2(p_0 + 1/p_0)$. We have

$$\exp\left(-\frac{1}{2}p_n^2\right) - \exp\left(-\frac{1}{2}p_{n+1}^2\right) = \int_{p_n}^{p_{n+1}} x \exp\left(-\frac{1}{2}x^2\right) dx$$

$$> (p_{n+1} - p_n) p_{n+1} \exp\left(-\frac{1}{2}p_{n+1}^2\right)$$

$$= (p_{n+1} - p_n) 2^{-n-1} p_0 \exp\left(-\frac{1}{2}p_0^2\right);$$

and summing over $n = 0, 1, \cdots$, we obtain

$$\exp\left(-\frac{1}{2}p_0^2\right) > p_0 \exp\left(-\frac{1}{2}p_0^2\right)\left(-\frac{1}{2}p_0 + \frac{1}{2}\sum_{n=1}^{\infty} 2^{-n} p_n\right),$$

$$\frac{1}{p_0} > -\frac{1}{2}p_0 + \frac{1}{2}\sum_{n=1}^{\infty} 2^{-n} p_n,$$

$$p_0 + \frac{1}{p_0} > \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n} p_n.$$

Remark 4. The "9" in the statement of the lemma can of course be improved. Numerical computation shows that it can be made less than 4.

Remark 5. For nonconvex V the corresponding weaker estimate is derived from other considerations [1, Thm. 2(a)]; let us give both estimates in comparable notation:

$$u = \frac{r^2}{32\sigma^2} - \frac{C_2}{10\sigma} - \frac{81}{32} \quad (\text{in } [1]);$$

$$u = \frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{\min(r, 1)}\right)^2 - \log 9 \quad (\text{here});$$

but

$$\log 9 < \frac{81}{32}, \quad \text{and} \quad \frac{1}{2}\left(\frac{r}{\sigma} - \frac{C_2}{r}\right)^2 \geqq \frac{r^2}{32\sigma^2} - \frac{C_2}{10\sigma}$$

always, and

$$\frac{1}{2}\left(\frac{r}{\sigma} - C_2\right)^2 \geqq \frac{r^2}{32\sigma^2} - \frac{C_2}{10\sigma} \quad \text{for} \quad \sigma C_2 \leqq \frac{1}{4};$$

these inequalities can be verified by reducing them to homogeneous quadratic inequalities with two variables by means of appropriate changes of variables.

Proof of Theorem 1 *in* [2]. Point (a) follows immediately from Lemma 1 in [2] and inequality (1); indeed, assuming that $\theta = 0$ and $\sigma = 1$ (the general case reduces to this one via a shift and a dilation), we obtain

$$\mathbf{P}\{\hat\theta(\xi) \in F_k\} = \gamma\{x: \hat\theta(x) \in F_k\} \leqq \gamma_V\{x: \hat\theta(x) \in F_k\} = \frac{1}{k!}\mathcal{M}_k(V)$$

$$\leqq \frac{1}{k!}\mathcal{M}_1^k(V) = \frac{1}{k!}C^k.$$

Let us prove point (b).

First step. From point (a) already proved, noting (2) we can derive that for any $r \geqq 1$

$$\mathbf{P}\{\hat\theta(\xi) \in F_k, \|\hat\theta(\xi) - \theta\| \leqq r\} \leqq \frac{1}{k!}\left(\frac{C_2 r}{\sigma}\right)^k;$$

$$\mathbf{P}\{\hat{\theta}(\xi)\in F_k\cup F_{k+1}\cup\cdots\cup F_n,\ \|\hat{\theta}(\xi)-\theta\|\leqq r\}\leqq\left(\frac{eC_2r}{k\sigma}\right)^k.$$

Indeed, assuming that $C_2r/(k\sigma)\leqq 1$ (otherwise there is nothing to prove), we have

$$\begin{aligned}\sum_{l=k}^{n}\frac{1}{l!}\left(\frac{C_2r}{\sigma}\right)^l &\leqq \min_{a\geqq 1} a^{-k}\sum_{l=k}^{\infty}\frac{1}{l!}\left(\frac{C_2r}{\sigma}a\right)^l\\ &\leqq \min_{a\geqq 1} a^{-k}\exp\left(\frac{C_2r}{\sigma}a\right)\\ &=\left(\min_{a\geqq 1} a^{-1}\exp\left(\frac{C_2r}{k\sigma}a\right)\right)^k=\left(\frac{eC_2r}{k\sigma}\right)^k.\end{aligned}$$

On the other hand,

$$\mathbf{P}\{\|\hat{\theta}(\xi)-\theta\|\geqq r\}\leqq 9\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)_+^2\right)$$

for $r\geqq 1$ by Lemma 4; thus,

$$\mathbf{P}\{\hat{\theta}(\xi)\in F_k\cup K_{k+1}\cup\cdots\cup F_n\}\leqq\left(\frac{eC_2r}{k\sigma}\right)^k+9\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)_+^2\right)$$

for any $r\geqq 1$.

Second step. Let us show that

$$\min_{r\geqq 1}\left(\frac{eC_2r}{k\sigma}\right)^k+9\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)_+^2\right)\leqq 10\left(\frac{e^2C_2^2}{k}\log\frac{k}{C_2^2}\right)^{k/2}$$

for $k>C_2^2$ and $\sigma^2k\log(k/C_2^2)\geqq 1$. For this we put $a=\sqrt{k}/C_2$ and take $r=\sigma\sqrt{k\log a^2}$. We then have

$$\left(\frac{eC_2r}{k\sigma}\right)^k=\left(\frac{e^2\log a^2}{a^2}\right)^{k/2},$$

$$\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)_+^2\right)=\exp\left(-\frac{k}{2}\left(\sqrt{2\log a}-\frac{1}{a}\right)_+^2\right).$$

It is not hard to check that

$$\frac{1}{2}\left(\sqrt{2\log a}-\frac{1}{a}\right)_+^2\geqq\log a-\frac{1}{2}\log(2\log a)-1$$

for all $a>1$; hence

$$\begin{aligned}\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)_+^2\right)&\leqq\exp\left(-k\left(\log a-\frac{1}{2}\log(2\log a)-1\right)\right)\\ &=\left(\frac{e^2\log a^2}{a^2}\right)^{k/2},\end{aligned}$$

which yields the required result.

Remark 6. If all the hypotheses of Theorem 1(b) are satisfied aside from the inequality $\sigma^2k\log(k/C_2^2)\geqq 1$, then we can use the estimate

$$\mathbf{P}\{\hat{\theta}(\xi)\in F_k\cup F_{k+1}\cup\cdots\cup F_n\}\leqq\left(\frac{eC_2}{k\sigma}\right)^k+9\exp\left(-\frac{1}{2}\left(\frac{1}{\sigma}-C_2\right)_+^2\right);$$

it is derived from the reasoning given above for $r=1$.

The next two lemmas (one is for the finite-dimensional case, the other for the infinite-dimensional case) are directed towards the proof of Theorem 3.

LEMMA 5. *Let $V\ (V\subset \mathbf{R}^n)$ be a convex compact set with piecewise C^2-smooth boundary and $\theta\in V$; then, for any $\sigma>0$ and $a>0$,*

$$\int \exp(a\Delta\varphi(x))\gamma_{\theta,\sigma}(dx)\leqq \exp\left(\frac{e^a}{\sigma}\mathcal{M}_1(V)\right)$$

(here Δ is the differential Laplace operator).

Proof. The Laplacian is the sum of the characteristic numbers of the second differential: $\Delta\varphi(x)=\lambda_1(x)+\cdots+\lambda_n(x)$ (note incidentally that this is the trace of the differential of the mapping $\hat{\theta}$, see (8)). Using the inequality $\exp(a\lambda)\leqq 1+(e^a-1)\lambda$, which is valid for $0\leqq\lambda\leqq 1$, and Lemma 2, we obtain

$$\int \exp(a\Delta\varphi(x))\gamma_{\theta,\sigma}(dx)\leqq \int \prod_{i=1}^{n}(1+(e^a-1)\lambda_i(x))\gamma_{\theta,\sigma}(dx)$$
$$\leqq \exp\left(\frac{e^a}{\sigma}\mathcal{M}_1(V)\right).$$

LEMMA 6. *Let $V\subset E_0$ be a convex GB-compact set and $\theta\in V$. Then, for any positive σ, δ and a,*

$$\int \exp\left(\frac{2a}{\delta^2}(\varphi_\delta(x)-\varphi(x))\right)\gamma_{\theta,\sigma}(dx)\leqq \exp\left(\frac{e^a}{\sigma}\mathcal{M}_1(V)\right),$$

where $\varphi_\delta(x)=\int\varphi(x+\delta y)\gamma(dy)$.

Proof. We can assume that V is a finite-dimensional convex solid with piecewise smooth boundary; the general case is derived from this one by means of an approximation from the inside. We shall assume that $(E,\gamma)=(\mathbf{R}^n,\gamma^n)$. We shall need the formula

$$\varphi_\delta(x)-\varphi(x)=\frac{1}{2}\int_0^{\delta^2}dt\int\gamma(dy)\,\Delta\varphi(x+\sqrt{t}y).$$

It can be checked by routine calculations (passage to polar coordinates in the integral with respect to y) or by application of Itô's formula to the stochastic differential $d\varphi(w_n(t))$, where w_n is an n-dimensional Wiener process. Let us use this formula, the convexity of the exponential function and Lemma 5:

$$\int \exp\left(\frac{2a}{\delta^2}(\varphi_\delta(x)-\varphi(x))\right)\gamma_{\theta,\sigma}(dx)$$
$$=\int\gamma_{\theta,\sigma}(dx)\exp\left(a\frac{1}{\delta^2}\int_0^{\delta^2}dt\int\gamma(dy)\Delta\varphi(x+\sqrt{t}y)\right)$$
$$\leqq\int\gamma_{\theta,\sigma}(dx)\frac{1}{\delta^2}\int_0^{\delta^2}dt\int\gamma(dy)\exp(a\Delta\varphi(x+\sqrt{t}y))$$
$$=\frac{1}{\delta^2}\int_0^{\delta^2}dt\int\gamma_{\theta,\sqrt{\sigma^2+t}}(dx)\exp(a\Delta\varphi(x))$$
$$\leqq\frac{1}{\delta^2}\int_0^{\delta^2}dt\exp\left(\frac{e^a}{\sqrt{\sigma^2+t}}\mathcal{M}_1(V)\right)\leqq\exp\left(\frac{e^a}{\sigma}\mathcal{M}_1(V)\right).$$

Proof of Theorem 3 *in* [2]. *First step.* As will be proved at the second step,

$$\varliminf_{\delta\to 0+}\frac{2}{\delta^2}(\varphi_\delta(x)-\varphi(x))\geqq K(\hat\theta(x))$$

for $\gamma_{\theta,\sigma}$-almost all $x\in E$. Because of this, point (a) reduces to Lemma 6:

$$\begin{aligned}\mathbf{E}\exp(aK(\hat\theta(\xi)))&\leqq\int\exp\left(a\varliminf_{\delta\to 0+}\frac{2}{\delta^2}(\varphi_\delta(x)-\varphi(x))\right)\gamma_{\theta,\sigma}(dx)\\&\leqq\varliminf_{\delta\to 0+}\int\exp\left(\frac{2a}{\delta^2}(\varphi_\delta(x)-\varphi(x))\right)\gamma_{\theta,\sigma}(dx)\\&\leqq\exp\left(\frac{e^a}{\sigma}\mathcal{M}_1(V)\right)=\exp\left(\frac{e^a}{\sigma}C\right).\end{aligned}$$

From point (a), for $a=\log(k\sigma/C)$ it follows that

$$\mathbf{P}\{K(\hat\theta(x))\geqq k\}\leqq\left(\frac{eC}{\sigma k}\right)^k;$$

and now it remains to apply Lemma 4 and the second step in the proof of Theorem 1.

Second step. Let us prove that

$$\varliminf_{\delta\to 0+}\frac{2}{\delta^2}(\varphi_\delta(x)-\varphi(x))\geqq K(\hat\theta(x))$$

for $\gamma_{\theta,\sigma}$-almost all $x\in E$. Let $K(\hat\theta(x))\geqq k$, $k<+\infty$. There exist a k-dimensional subspace $H_k\subset E_0$ and an $\varepsilon>0$ such that $\hat\theta(x)+\eta\in V$ for all $\eta\in H_k$ such that $\|\eta\|\leqq\varepsilon$. Consider the orthogonal projector $P_k: E_0\to H_k$ and extend it to a measurable linear operator $P_k: E\to H_k$. For $\eta\in H_k$ put $Q(\eta)=\eta$ if $\|\eta\|\leqq\varepsilon$ and $Q(\eta)=0$ if $\|\eta\|>\varepsilon$; for $y\in E$ put $Q(y)=Q(P_k(y))$. We recall the definition of φ, note that $\hat\theta(x)+Q(\delta y)\in V$, drop the odd terms with respect to y (their integral is zero), and discard inessential measurements:

$$\begin{aligned}\varphi_\delta(x)-\varphi(x)&=\int(\varphi(x+\delta y)-\varphi(x))\gamma(dy)\\&\geqq\int(\langle\hat\theta(x)+Q(\delta y),x+\delta y\rangle-\frac{1}{2}\|\hat\theta(x)+Q(\delta y)\|^2\\&\quad-\langle\hat\theta(x),x\rangle+\tfrac{1}{2}\|\hat\theta(x)\|^2)\gamma(dy)\\&=\int\left(\langle Q(\delta y),\delta y\rangle-\frac{1}{2}\|Q(\delta y)\|^2\right)\gamma(dy)\\&=\int_{H_k}\left(\langle Q(\delta\eta),\delta\eta\rangle-\frac{1}{2}\|Q(\delta\eta)\|^2\right)\gamma^k(d\eta)\\&=\frac{\delta^2}{2}\int_{\|\eta\|\leqq\varepsilon/\delta}\|\eta\|^2\gamma^k(d\eta);\end{aligned}$$

thus,

$$\begin{aligned}\varliminf_{\delta\to 0+}\frac{2}{\delta^2}(\varphi_\delta(x)-\varphi(x))&\geqq\lim_{\delta\to 0+}\int_{\|\eta\|\leqq\varepsilon/\delta}\|\eta\|^2\gamma^k(d\eta)\\&=\int_{H_k}\|\eta\|^2\gamma^k(d\eta)=k.\end{aligned}$$

The next two lemmas pertaining to the finite-dimensional case set the stage for the proof of Theorem 2.

LEMMA 7. *Suppose that the quadratic form A on $\mathbf{R}^n$ has characteristic values $\lambda_1 \geqq \cdots \geqq \lambda_n \geqq 0$. Then, for any $m \in \mathbf{R}^n$, $\sigma \in (0, 1/\sqrt{2\lambda_1})$,*

$$\int \exp A(m+\sigma x)\gamma^n(dx) \leqq \left(\prod_{i=1}^n (1-2\sigma^2\lambda_i)\right)^{-1/2} \exp \frac{A(m)}{1-2\sigma^2\lambda_1}.$$

Proof. We can assume that A has been reduced to the principal axes: $A(x_1, \cdots, x_n) = \Sigma\lambda_i x_i^2$. Then the integral decomposes into the product of one-dimensional integrals that are easy to compute:

$$\int_{-\infty}^{+\infty} \exp\left(\lambda_i(m_i+\sigma x_i)^2\right)\gamma^1(dx_i) = (1-2\lambda_i\sigma^2)^{-1/2} \exp\left(\frac{\lambda_i m_i^2}{1-2\lambda_i\sigma^2}\right).$$

LEMMA 8. *Let $V (V \subset \mathbf{R}^n)$ be a convex compact set, $0 \subset V$, and $\|\theta\| \leqq r$ for all θ in V. Then, for any $a, b > 0$, $c \in (0, \frac{1}{2})$,*

$$\iint \exp\left(c\left(\frac{\|\hat\theta(ax+by) - \hat\theta(ax-by)\|}{2b}\right)^2\right)\gamma^n(dx)\gamma^n(dy)$$
$$\leqq \exp\left(\frac{\mathcal{M}_1(V)}{a\sqrt{1-2c}} + \frac{cb^2r^2}{a^4(1-2c)}\right).$$

Proof. We can assume that V is a convex solid with piecewise smooth boundary. We use formulas (5) and (8):

$$\begin{aligned}\|\hat\theta(ax+by) - \hat\theta(ax-by)\|^2 &\leqq \langle\hat\theta(ax+by) - \hat\theta(ax-by), 2by\rangle \\ &= D\varphi(ax+by, 2by) - D\varphi(ax-by, 2by) \\ &= 2\frac{d}{dt}\varphi(ax+tby)\Big|_{t=-1}^{t=+1} \\ &= 2\int_{-1}^{+1} \frac{d^2}{dt^2}\varphi(ax+tby)\,dt \\ &= 2\int_{-1}^{+1} D^2\varphi(ax+tby, by)\,dt.\end{aligned}$$

We take into account the convexity of the exponential function:

$$\begin{aligned}&\iint \exp\left(c\left(\frac{\|\hat\theta(ax+by) - \hat\theta(ax-by)\|}{2b}\right)^2\right)\gamma^n(dx)\gamma^n(dy) \\ &\leqq \iint \exp\left(\frac{c}{b^2}\frac{1}{2}\int_{-1}^{+1} D^2\varphi(ax+tby, by)\,dt\right)\gamma^n(dx)\gamma^n(dy) \\ &\leqq \iint \frac{1}{2}\int_{-1}^{+1} \exp\left(\frac{c}{b^2}D^2\varphi(ax+tby, by)\right)dt\,\gamma^n(dx)\gamma^n(dy).\end{aligned}$$

Changing the order of integration, we obtain $\frac{1}{2}\int_{-1}^{+1} J(t)\,dt$, where

$$J(t) = \iint \exp\left(cD^2\varphi(ax+tby, y)\right)\gamma^n(dx)\gamma^n(dy).$$

It suffices to prove that, for all $t \in [-1, 1]$,

$$J(t) \leqq \exp\left(\frac{\mathcal{M}_1(V)}{a\sqrt{1-2c}} + \frac{cb^2r^2}{a^4(1-2c)}\right).$$

We cannot apply Lemma 7 to the integral for $J(t)$ directly due to the fact that y appears in the expression $ax+tby$. We switch from the variables x, y to the new variables $v=(ax+tby)s^{-1}$, $w=(tbx-ay)s^{-1}$, where $s=(a^2+t^2b^2)^{1/2}$, and then $\gamma^n(dx)\gamma^n(dy)=\gamma^n(dv)\gamma^n(dw)$. We apply Lemma 7 to the integral with respect to w and note that the characteristic values $\lambda_i(x)$ of the quadratic form $D^2(x,h)$ do not exceed one:

$$J(t)=\iint \exp(cD^2\varphi(sv,(tbv-aw)s^{-1}))\gamma^n(dv)\gamma^n(dw)$$
$$\leq \int \left(\prod_{i=1}^n \left(1-2\left(\frac{a}{s}\right)^2 c\lambda_i(sv)\right)\right)^{-1/2} \exp\left(\frac{cD^2\varphi(sv, tbvs^{-1})}{1-2(a/s)^2c}\right)\gamma^n(dv).$$

Note that $D^2\varphi(sv,sv)\leq r^2$. Indeed, if $sv\in V$, then $D^2\varphi(sv,sv)=\|sv\|^2\leq r^2$; but if $sv\notin V$, then the quadratic form $D^2\varphi(sv,h)$ is degenerate in the direction of the vector $h=sv-\hat\theta(sv)$, and therefore $D^2\varphi(sv,sv)=D^2\varphi(sv,\hat\theta(sv))\leq\|\hat\theta(sv)\|^2\leq r^2$. Using this estimate, the routine inequality

$$(1-\delta\lambda)^{-1/2}\leq 1+\left(\frac{1}{\sqrt{1-\delta}}-1\right)\lambda \quad \text{for} \quad \lambda\in[0,1], \quad \delta=\frac{2a^2c}{s^2}<1,$$

as well as Lemma 2 for $z=1/\sqrt{1-\delta}-1$, we obtain

$$J(t)\leq \exp\left(\frac{ct^2b^2r^2}{s^4(1-\delta)}\right)\int\left(\prod_{i=1}^n(1+z\lambda_i(sv))\right)\gamma^n(dv)$$
$$\leq \exp\left(\frac{ct^2b^2r^2}{s^4(1-\delta)}\right)\exp\left(\frac{\mathcal{M}_1(V)}{s\sqrt{1-\delta}}\right)$$
$$\leq \exp\left(\frac{\mathcal{M}_1(V)}{a\sqrt{1-\delta}}+\frac{cb^2r^2}{a^4(1-\delta)}\right);$$

it remains to note that $\delta\leq 2c$.

Proof of Theorem 2 in [2]. Let us first prove point (a). We can assume that $\theta=0$. The correlated vectors ξ_1 and ξ_2 can be represented in the form $\xi_1=a\zeta_1+b\zeta_2$, $\xi_2=a\zeta_1-b\zeta_2$, where $a=\sigma\sqrt{(1+\rho)/2}$, $b=\sigma\sqrt{(1-\rho)/2}$, and ζ_1 and ζ_2 are independent random vectors with standard Gaussian distribution. Let us apply Lemma 8 for $c=\frac14$:

$$\mathbf{E}\exp\left(\frac{\|\hat\theta(\xi_1)-\hat\theta(\xi_2)\|}{4b}\right)^2=\iint\exp\left(\frac{\|\hat\theta(ax+by)-\hat\theta(ax-by)\|}{4b}\right)^2\gamma^n(dx)\gamma^n(dy)$$
$$\leq\exp\left(\frac{\sqrt2 C}{a}+\frac{b^2r^2}{2a^4}\right).$$

But $r\leq\sqrt{2\pi}C$, and so

$$\frac{\sqrt2C}{a}+\frac{b^2r^2}{2a^4}\leq\frac{C}{a}\left(\sqrt2+\frac{\pi b^2C}{a^3}\right)$$
$$=\frac{C}{a}\left(\sqrt2+\frac{\pi\sqrt2C}{\sigma(1+\rho)^{3/2}}(1-\rho)\right)$$
$$\leq\frac{C}{a}\left(\sqrt2\,\frac{\pi\sqrt2C}{\sigma(\frac32)^{3/2}}\,\frac{\sigma}{5C}\right)\leq 1.9\frac{C}{a}.$$

Hence

$$\mathbf{P}\left\{\frac{\|\hat\theta(\xi_1)-\hat\theta(\xi_2)\|}{\sqrt{\sigma(1-\rho)}}\geqq 3\sqrt{\sigma u+2C}\right\}$$

$$\leqq \exp\left(-\left(\frac{\sqrt{\sigma(1-\rho)}3\sqrt{\sigma u+2C}}{4b}\right)^2\right)\mathbf{E}\exp\left(\left(\frac{\|\hat\theta(\xi_1)-\hat\theta(\xi_2)\|}{4b}\right)^2\right)$$

$$\leqq \exp\left(-\frac{9\sigma(1-\rho)(\sigma u+2C)}{16b^2}+\frac{1.9C}{a}\right)$$

$$=\exp\left(-\frac{9}{8}u-\frac{9C}{4\sigma}+\frac{1.9\sqrt{2}C}{\sigma\sqrt{1+\rho}}\right)$$

$$\leqq \exp\left(-u-\frac{C}{\sigma}\left(\frac{9}{4}-\frac{1.9\sqrt{2}}{\sqrt{1+\rho}}\right)\right)\leqq \exp(-u).$$

This proves point (a). Substituting $u=q^2/(9\sigma)-2C/\sigma$, we rewrite it as

$$\mathbf{P}\left\{\frac{\|\hat\theta(\xi_1)-\hat\theta(\xi_2)\|}{\sqrt{\sigma(1-\rho)}}\geqq q\right\}\leqq \exp\left(-\frac{q^2}{9\sigma}+\frac{2C}{\sigma}\right).$$

In order to prove point (b), we apply the last inequality to the set $V\cap B(\theta, r)$; using Lemma 4 and (2) we have, for any $r\geqq 1$,

$$\mathbf{P}\left\{\frac{\|\hat\theta(\xi_1)-\hat\theta(\xi_2)\|}{\sqrt{\sigma(1-\rho)}}\geqq q\right\}$$

$$\leqq \mathbf{P}\left\{\frac{\|\hat\theta(\xi_1)-\hat\theta(\xi_2)\|}{\sqrt{\sigma(1-\rho)}}\geqq q,\ \|\hat\theta(\xi_1)-\theta\|\leqq r,\ \|\hat\theta(\xi_2)-\theta\|\leqq r\right\}$$

$$+\mathbf{P}\{\|\hat\theta(\xi_1)-\theta\|\geqq r\}+\mathbf{P}\{\|\hat\theta(\xi_2)-\theta\|\geqq r\}$$

$$\leqq \exp\left(-\frac{q^2}{9\sigma}+\frac{2C_2r}{\sigma}\right)+18\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)_+^2\right).$$

It suffices to choose $r\geqq 1$ such that

$$-\frac{q^2}{9\sigma}+\frac{2C_2r}{\sigma}\leqq -(u+3)\quad\text{and}\quad -\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)_+^2\leqq -(u+3),$$

for $\exp(-u-3)+18\exp(-u-3)<\exp(-u)$. Take $r=\max(1, \sigma(\sqrt{2(u+3)}+C_2))$; it is not hard to check that for $r=\sigma(\sqrt{2(u+3)}+C_2)$ we have $q=3\sqrt{\sigma}(\sqrt{u+3}+\sqrt{2C_2})$ (the first case), while for $r=1$ we have $q=3\sqrt{\sigma(u+3)+2C_2}$ (the second case). In the first case we simply have $-q^2/(9\sigma)+2C_2r/\sigma=-(u+3)$ and $-\frac{1}{2}(r/\sigma-C_1)^2=-(u+3)$. In the second case we again have $-q^2/(9\sigma)+2C_2r/\sigma=-(u+3)$ and, moreover,

$$-\frac{1}{2}\left(\frac{r}{\sigma}-C_2\right)^2=-\frac{1}{2}\left(\frac{1}{\sigma}-C_2\right)^2\leqq -(u+3),$$

since $\sigma(\sqrt{2(u+3)}+C_2)\leqq 1$.

Proof of Theorem 4 *in* [2]. For a finite-dimensional V, the desired assertions follow from Theorem 2. The passage to the limit for the general case is carried out via Remark 3 in [1].

REFERENCES

[1] B. S. Tsirel'son, *A geometric approach to a maximum likelihood estimation for infinite-dimensional location*. I, Theory Probab. Appl., 27 (1982), pp. 411-418.

[2] ———, *A geometric approach to a maximum likelihood estimation for infinite-dimensional location*. II, Theory Probab. Appl., 30 (1985), pp. 820-828.

[3] S. Chevet, *Processus Gaussiens et volumes mixtes*, Z. Wahrsch. Verw. Gebiete, 36 (1976), pp. 47-65.

[4] Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Nauka, Leningrad, 1980. (In Russian.)