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DYNAMIC ROUTING IN POLLING SYSTEMS

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Polling systems have important applications to Telecommunication Systems, Data Link Protocols, Local Area Computer Networks, Flexible Manufacturing Systems, etc. As such they have been the subject of extensive research in recent years, most notably in the context of cyclic queues. The focus of much of this research has been on evaluating performance measures of fixed-template routing schemes with different service disciplines, namely the Exhaustive, Gated and Limited Service policies. Analyses have concentrated on obtaining equilibrium mean-value or approximate results for various policies, and only recently have pseudo-conservation laws been derived.
In this paper, optimal routing policies are derived that lead to adaptive dynamic control of the server's polling schemes for both the Exhaustive and Gated service disciplines with or without switching times between channels. These dynamic routing rules utilize current system information at suitably defined decision epochs, they are surprisingly simple, and of a form amenable to direct engineering implementation.

1. INTRODUCTION

Cyclic queues is the name given to a specific queueing network. In its most basic form it is a system composed of K channels, where customers arrive at each channel independently via a random stream. There is one server in the system who moves from channel to channel in a "cyclic fashion", i.e., the server stays at channel j ($j = 1, \dots, K$) for a length of time determined by the queue discipline and then moves to channel $j+1$. When the server completes channel K he must revert to channel 1, hence the name cyclic. Recently, as computer scientists continue to make use of queueing theory to design efficient systems, the name "Polling Systems" has arisen to describe cyclic single server networks, as has the "Token Ring" description. Cyclic queues have usually been studied under the assumption of Poisson arrivals, although the study of discrete-time polling systems utilizes other arrival schemes (e.g., Kleinrock and Levy [1], (1987)). The j^{th} channel is characterized by a general service time distribution function $G_j(x)$, which is assumed degenerate at $x = 1$ in slotted-time models. Although Cooper [2] (1970) was the first to explicitly name and study cyclic queues, the origins of these systems and the methods of analysis appropriate to them are found in (alternating) priority queues and queues with server interruptions.

The queue disciplines most often studied in the context of cyclic queues are the Exhaustive, Gated, and Limited Service. The Exhaustive discipline requires that the server stay and service channel j until the moment that the server becomes idle at j, (i.e., until channel j is empty) whereupon the server moves on to channel $j+1$, serving it exhaustively. If there is zero switching time or set-up times between channels, then for $K=2$ the system is obviously an "Alternating Priority Queue" (Avi Itzhak et al [3] (1965), and Takacs [4] (1968)). The case $K=1$ where there is non-zero switching or set-up time is a queueing system whose server goes on vacation (Levy and Yechiali [5] (1975)).

The Gated discipline requires the server to service only those customers present in channel j upon his arrival to channel j, i.e., the server "gates off" the customers he sees upon his arrival, serves them and then moves on to channel $j+1$. The Limited Service discipline specifies for the server visiting channel j to serve either 1, at most k_j, or to deplete the channel by one customer. As cyclic queues are so important in applications, for example in Telecommunications systems, Data Link protocols, Local Area computer networks, Flexible Manufacturing systems, etc., they have been the subject of many recent papers and at least one monograph (Takagi [6] (1986)). However, a study of the literature reveals that much of the research deals with only mean-value or approximate analysis of different performance criteria, (see, for example, Watson [7] (1984), Eisenberg [8] (1972), and Halfin [9] (1975)). Moreover cyclic queues have until recently been studies only under a *fixed* cyclic ordering, and only in special cases (e.g., identical service or arrival rates) do exact results seem to be known for waiting times during a cycle. Recently Boxma and Groenendijk [10] (1986) have derived "Pseudo-conservation" laws. To our knowledge, as of this writing, no studies of optimization within cyclic queues have been published.

In this paper we explore optimization of various performance criteria for a cyclic-type queueing system. The control, or decision variable, is the choice by the server, at properly defined decision epochs, of which channels to service over the decision horizon. We study this control problem for both the Exhaustive and Gated disciplines under 2 scenarios:

1. Zero switch times: The server upon completion of service to a channel moves instantaneously to another channel.
2. Non-zero switch times between all channels: If the decision is made by the server to switch from channel i to channel j, a switch time S_{ij} occurs between completion of service in i and commencement of service in j, where S_{ij} is a suitably defined random variable.

In section 2 we analyze the problem of controlling the server's path so as to minimize weighted waiting times of the customers. We study the Exhaustive and the Gated disciplines with zero switching times, formulate both models as Markov Decision processes, and derive the optimality equations for controlling the system under each regime. However, the curse of dimensionality has precluded us, so far, from obtaining "nice" and easily implementable operating rules similar to the "Right of Way" policies derived by Meilijson and Yechiali [11] (1977). Nevertheless, the optimality equations allow for numerical computations of optimal polling policies.

We consider next (Section 3) optimization of the server's "greedy" instincts: the server wishes to choose the path to minimize cycle time, thus, implicitly, reducing waiting times. Of course, if the server can choose this path, then the system is no longer a cyclic queue. We name it a "Pseudo-Cyclic" Queue. This corresponds to a *dynamic priority rule* for choosing channels. The rules derived are of a *surprisingly simple form*, and thus constitute an *adaptive control* procedure amendable to direct implementation within the system.

2. MINIMIZING WEIGHTED WAITING TIMES

2.1 The Exhaustive Regime

Recall that we deal with K channels, where the Poissonian arrival rate to channel i is $\lambda_i, i = 1, \ldots, K$. Each type i customer brings along a random service requirement distributed as V_i, where V_i has probability distribution function $G_i(\cdot)$.

Consider first the Exhaustive regime. The server, having chosen to serve a specific channel, say j, must stay there and service all customers at that channel exhaustively, i.e., until channel j is empty, whereupon the server must decide which channel to service next. Define the state of the system at any point in time, t, as

$$\underline{Q}(t) = \{Q_1(t), \ldots, Q_K(t)\}$$

where $Q_i(t)$ = number of customers in channel i at t. *Decision epochs* occur when the server has completed one channel, and now must move to some channel with a positive number of occupants, as idleness is not permitted. The State Space is therefore I^K where $I = \{0, 1, 2, \ldots\}$, and the Action Space is $A = \{1, \ldots, K\}$ with generic element a, where if t is a decision epoch, then $a(t) = i$ simply means the server has chosen to service channel i next.

By looking at the system only at decision moments (channel completions), $\{t_i\}_{i=1}^{\infty}$, we can define an *embedded Markov chain* over the system at these points. Letting $a(t_i) = a_i$ = action taken at i^{th} transition, we have

$$P(\underline{Q}(t_i)\varepsilon S|\underline{Q}(t_{i-1}), a_{i-1}; Q(t_{i-2}), a_{i-2}; \ldots; Q(t_0), a_0) = P(\underline{Q}(t_i)\varepsilon S|\underline{Q}(t_{i-1}), a_{i-1})$$

where $S \subset I^K$, and a_i = channel served during $(t_{i+1} - t_i)$. Under the stability assumption,

$$\sum_{i=1}^{K} \lambda_i EV_i \triangleq \sum_{i=1}^{K} \rho_i < 1 ,$$

the t_i are well defined stopping times.

Let $a_i = j$, and say $a_{i-1} = \ell$, and for ease of exposition, let $\ell < j$, then $\underline{Q}(t_i) = \{n_1, \ldots, n_{\ell-1}, 0, n_{\ell+1}, \ldots, n_j, \ldots, n_K\}$, where obviously $Q_r(t_i) = n_r$. The server must now service channel j exhaustively; as there are n_j occupants in channel j at transition time t_i, it is obvious that the server will stay at channel j for a random time distributed as the sum of n_j $M/G/1$ busy periods of type j. Suppose $a_i = j$, and let $X_j \equiv t_{i+1} - t_i$ = "Occupation time of the server at the channel chosen at the i^{th} transition". It is clear that X_j is a function of n_j, G_j, λ_j, and it is known (Cooper [12] (1981) Pg. 231, Takacs [13] (1967) Pg. 109) that

$$P(X_j \leq t) = \sum_{n=n_j}^{\infty} \frac{n_j}{n} \int_0^t e^{-\lambda_j x} \frac{(\lambda_j x)^{n-n_j}}{(n-n_j)!} dG_j^{*n}(x) \equiv F_{X_j}(t) \tag{1}$$

where $G^{*n}(\cdot)$ denotes the n^{th} convolution of G.

As X_j is the sum of n_j ordinary $M/G_j/1$ busy periods it readily follows that its Laplace Transform is given by

$$E(e^{-sX_j}) \triangleq \widetilde{X}_j(s) = (\widetilde{B}_j(s))^{n_j} \triangleq \widetilde{V}_j(s + \lambda_j - \lambda_j \widetilde{B}(s))^{n_j} \tag{2}$$

where B_j is a random variable distributed as an ordinary $M/G/1$ busy period of type j, and $\widetilde{B}_j(s)$ is its Laplace transform. Equation (2) leads directly to

$$E(X_j) = \frac{n_j E(V_j)}{1-\rho_j} \tag{3}$$

$$E(X_j^2) = \frac{n_j}{(1-\rho_j)^2}\left[\frac{E(V_j^2)}{1-\rho_j} + (n_j - 1)(EV_j)^2\right] . \tag{4}$$

We may define the one-step transition probabilities for our system quite simply; let

$$\underline{Q}(t_i) = \underline{Q}, \qquad \text{and} \qquad \underline{Q}(t_{i+1}) = \underline{Q}' ,$$

where

$$Q_j(t_i) = n_j \qquad \text{and} \qquad Q_j(t_{i+1}) = n_j' .$$

Then

$$P_{\underline{Q},\underline{Q}'}(a) = P(\underline{Q}(t_{i+1}) = \underline{Q}' | \underline{Q}(t_i) = \underline{Q}, a_i = a) \tag{5}$$

Of course, to have defined an embedded Markov chain over the system means that in reality, $\underline{Q}$ is a semi-Markov process with kernel

$$P(\underline{Q}', t_{i+1} - t_i \le t | \underline{Q}, a) = \int_0^t \exp\left[-(\sum_{j\neq a} \lambda_j)x\right] \prod_{j\neq a} \frac{(\lambda_j x)^{n_j'-n_j}}{(n_j'-n_j)!} dF_{X_a^{(x)}} \tag{6}$$

since obviously $(t_{i+1} - t_i | a_i = a) = X_a$, and for our process, if $\sum_{j=1}^{K} \rho_j < 1$, $P(t_{i+1} - t_i < \infty) = 1 \ \forall i$. Equation (5) is therefore simply equation (6) evaluated at $t = \infty$, i.e.,

$$P_{\underline{Q},\underline{Q}'}(a) = \int_0^\infty e^{-(\sum_{j\neq a} \lambda_j)x} \prod_{j\neq a} \frac{(\lambda_j x)^{n_j'-n_j}}{(n_j'-n_j)!} dF_{X_a}(x) \tag{7}$$

is the set of one-step transition probabilities if we take action a in state $\underline{Q}$. Note that to apply (semi)-Markov Decision Process theory to our problem, we need the one-step transition probabilities, equation (7), and their associated "one stage costs", which are defined only in the context of a performance criterion to be optimized. So, as one example, say we want to minimize the expected (average) cost of running the system by choosing a proper operating rule. A sufficient condition for an optimal policy to exist is that the semi-Markov process, $\underline{Q}(t)$, be regenerative, with finite expected length, (Ross [14] (1970), Pg. 159). However as $\underline{Q}$ is a recurrent state ($\sum_{j=1}^{K} \rho_j < 1$) this is obviously satisfied.

To get the costs associated with a transition, assume linear holding costs in that the system is charged \$ C_i per unit time for each type i customer waiting in the queue. Therefore costs incurred per transition from state $\underline{Q}$ to $\underline{Q}'$ are:

1. Costs incurred to channel a (a is the serviced channel)
2. Costs incurred to channels i, $i \neq a$.

The costs incurred to channel i, $i \neq a$ are

a) The cost of holding the *initial* n_i during X_a, and
b) The waiting cost of arrivals during X_a.

Now, if $m_i = n_i' - n_i$ = number of new arrivals to i during X_a, then by Poissonian arrivals it is obvious that $E(m_i) = \lambda_i E(X_a)$, and

$$\begin{aligned}&\text{Expected costs in } i = C_i \cdot (\text{total expected waiting time in } i \text{ during } X_a) \\ &= C_i \cdot (\text{expected waiting time of original } n_i + \text{ expected waiting time of new arrivals})\ .\end{aligned} \tag{8}$$

Following Yechiali [15] (1976), by Poisson arrivals the expected total wait of new arrivals can be written as the product:

$$E(\text{wait of an } \textit{arbitrary} \text{ new arrival}) \cdot E(\text{number of new arrivals}),$$

and as shown there, the expected waiting time of any arbitrary arrival during X_a is $\frac{E(X_a^2)}{2E(X_a)}$, the random modification of X_a. Therefore equation (8) is

$$= C_i(n_i E(X_a) + \lambda_i E(X_a)\frac{E(X_a^2)}{2E(X_a)}) = C_i(n_i E(X_a) + \frac{\lambda_i}{2}E(X_a^2)) \tag{9}$$

By equations (3) and (4), total costs incurred to the system besides those in channel a during X_a is

$$\sum_{i \neq a}\left\{C_i n_i \frac{n_a E(V_a)}{1-\rho_a} + \frac{C_i \lambda_i}{2}\frac{n_a}{(1-\rho_a)^2}\left[\frac{E(V_a^2)}{1-\rho_a} + (n_a - 1)(EV_a)^2)\right]\right\} \tag{10}$$

As for the costs incurred in channel a while servicing it, we have total expected cost of serving channel a exhaustively

$$\begin{aligned}&= C_a \cdot E(\text{wait of original } n_a) \\ &+ C_a \cdot E(\text{number of } a\text{−arrivals during } X_a) \cdot E(\text{wait of each } a\text{−arrival})\end{aligned} \tag{11}$$

Recalling that the expected number served during a regular busy period in an $M/G/1$ queue is $\frac{1}{1-\rho}$, we immediately see that E(number served during X_a)$= n_a/(1-\rho_a)$. In general, from the concept of delayed busy periods, or server vacation, we can get the average waiting time of a customer who *arrived* (and obviously was served) during a busy period that started with n initial customers, i.e., simply treat the period spent serving the original n customers as the delay, or vacation period (U in Levy and Yechiali [5] (1976)), and the remaining part as the (delayed) busy period.

Recall equation (38) in Levy & Yechiali [5] (1976)

$$E(W) = \frac{\lambda E(V^2)}{2(1-\rho)} + \frac{E(U^2)}{2E(U)} + EV \tag{12}$$

$= E$ (wait incurred by a customer arriving during a "busy cycle")
where V is the service time, λ the arrival rate, and $\rho = \lambda EV$.

To apply (12), simply realize that $U = \sum_{m=1}^{n} V_m =$ service of original n customers, where $V_m \sim V$. Hence,

$$EU = nE(V), \text{ and } E(U^2) = nE(V^2) + n(n-1)(EV)^2 \tag{13}$$

Now for our case,

$$\begin{aligned} &E(\text{waiting time for each type } a \text{ arriving and serviced during } X_a) = \\ &= \frac{\rho_a}{1-\rho_a}\frac{E(V_a^2)}{2E(V_a)} + \frac{n_aE(V_a^2) + n_a(n_a-1)(EV_a)^2}{2n_aE(V_a)} + E(V_a) \\ &= \frac{1}{1-\rho_a}\frac{E(V_a^2)}{2E(V_a)} + \frac{n_a+1}{2}E(V_a) \end{aligned} \tag{14}$$

so that the total wait of the new arrivals to channel a is

$$\begin{aligned} &\frac{\rho_a n_a}{1-\rho_a}\left[\frac{1}{1-\rho_a}\frac{E(V_a^2)}{2E(V_a)} + \frac{n_a+1}{2}E(V_a)\right] \\ &= \frac{\lambda_a n_a E(V_a^2)}{2(1-\rho_a)^2} + \frac{\rho_a n_a(n_a+1)}{2(1-\rho_a)}E(V_a) , \end{aligned} \tag{15}$$

as the number of expected new arrivals to channel a during X_a is $\frac{\rho_a n_a}{1-\rho_a}$.

The expected waiting time for the *original* n_a customers is obviously

$$\frac{n_a(n_a+1)}{2}E(V_a) . \tag{16}$$

Therefore, the expected total waiting time incurred at channel a during X_a is (15) +(16)

$$= \frac{\lambda_a n_a E(V_a^2)}{2(1-\rho_a)^2} + \frac{n_a(n_a+1)E(V_a)}{2(1-\rho_a)} . \tag{17}$$

Using (17) and (10), we can finally write the total *expected* cost incurred by a transition from $\underline{Q}$ during an "exhaustive sojourn" at channel a, as

$$\begin{aligned} C(\underline{Q}, a) = \{\sum_{i\neq a} C_i[n_i\frac{n_aE(V_a)}{1-\rho_a} + \frac{\lambda_i}{2}\frac{n_a}{(1-\rho_a)^2}[\frac{E(V_a^2)}{1-\rho_a} + (n_a-1)(EV_a)^2]] \\ + C_a(\frac{\lambda_a n_a E(V_a^2)}{2(1-\rho_a)^2} + \frac{n_a(n_a+1)E(V_a)}{2(1-\rho_a)})\} \end{aligned} \tag{18}$$

Equations (18), (7) and (1) are the necessary ingredients for formulating the optimality equation of this semi-Markov decision process. (See equation (24) below).

2.2 The Gated Regime

By similar reasoning, we can get the law of motion and one-step costs for the *gated* regime. Let Y_a denote the server occupation time in channel a under the *gated* regime. Then, if $\underline{Q}(t)$ is as above,

$$F_{Y_a}(y) = P(Y_a \leq y) = P(\sum_{i=1}^{n_a} V_{ai} \leq y) = G_a^{*n_a}(y) \tag{19}$$

where $V_{ai} \sim V_a$. Obviously, $\widetilde{Y}_a(s) = [\widetilde{V}_a(s)]^{n_a}$ so that

$$\begin{aligned} E(Y_a) &= n_a E(V_a) \\ E(Y_a^2) &= n_a(n_a - 1)(E(V_a))^2 + n_a E(V_a^2) \ . \end{aligned} \tag{20}$$

Our one-step transition probabilities are now

$$\Omega_{\underline{Q},\underline{Q}'}(a) = \int_0^\infty e^{-(\sum_{i=1}^K \lambda_i)y} \prod_{i=1}^K \frac{(\lambda_i y)^{r_i}}{r_i!} dF_{Y_a}(y) \tag{21}$$

where $\underline{Q} = (n,\ldots,n_k)$, $\underline{Q}' = (n_1 + r_1, \ldots, n_{a-1} + r_{a-1}, r_a, n_{a+1} + r_{a+1}, \ldots, n_k + r_k)$. Now the expected costs incurred to channel i, $i \neq a$ (see equation (9)) are

$$\begin{aligned} &C_i n_i E(Y_a) + \frac{C_i \lambda_i}{2} E(Y_a^2) \\ &= C_i n_i n_a E(V_a) + \frac{C_i \lambda_i}{2} \left(n_a(n_a - 1)(E(V_a))^2 + n_a E(V_a^2)\right) \ , \end{aligned} \tag{22}$$

and the expected costs in a are

$$C_a \left(\frac{n_a(n_a + 1)}{2} E(V_a) + \frac{\lambda_a}{2}[n_a(n_a - 1)(E(V_a))^2 + n_a E(V_a^2)]\right) \ . \tag{23}$$

2.3 The Optimality Equations

Denoting the sum of equations (22) and (23) as $\Gamma(\underline{Q}, a)$, we may write the optimality equations for the *exhaustive* and *gated* regimes, respectively, as (see Ross [14] (1970) Theorem 7.6)

$$h(\underline{Q}) = \min_a \left(C(\underline{Q}, a) + \sum_{\underline{Q}'} P_{\underline{Q},\underline{Q}'}(a) h(\underline{Q}') - gE(X_a) \right) \tag{24}$$

and

$$\psi(\underline{Q}) = \min_a \left(\Gamma(\underline{Q}, a) + \sum_{\underline{Q}'} \Omega_{\underline{Q},\underline{Q}'}(a) \psi(\underline{Q}') - \beta E(Y_a) \right) \tag{25}$$

where g and β are the average minimal costs incurred by the system, per unit time, under the exhaustive and gated regimes, respectively.

In principle, equations (24) and (25) may be solved for the optimal policies to minimize weighted waiting times. However, a glance at the transition probabilities is sufficient to conclude that this would be a formidable task. Difficulties of this sort have precluded previous investigators from solving such problems. However, in what follows, we will be able to get simple and elegant optimal routing rules if we approach the problem via an *alternative* procedure. Rather than investigating ways to minimize the sum of the individual waiting times, which is notoriously difficult, we focus on methods of routing the server so as to optimize a system objective. We also seek rules which are amenable to direct engineering implementation. To that end, we first truncate the decision horizon to a *single cycle* to exploit the nature of 'Polling', in that we propose a class of policies, "Pseudo Cyclic", in which the server must complete a "Tour" (rather than a cycle) of the 'unserved' channels before returning to any 'served' ones; a "Tour" being any path, or route that is a permutation of the labels $(1, \ldots, K)$. There is a degree of fairness that is incorporated into this class of policies in that channels may not be overlooked during a tour, but yet the server may still optimize the system performance within a tour.

Many possible system objectives may be dealt with in this manner, some possibilities are:

1. minimizing the length of each tour.
2. maximizing the number of customers served during each tour,
3. minimizing the number of customers in the system at the termination of each tour.

Note that in a system with zero switching times, the length of the busy period is unaffected by any service policy as long as the policy disallows idleness. However, when switching times are positive, the policy associated with any objective will determine the length of this busy period. These issues must be studied and evaluated in the system design.

We devote our attention for the remainder of this paper towards objective 1, the minimization of 'pseudo cycle', or 'tour' times.

3. MINIMIZING CYCLE TIME

3.1 Exhaustive Regime, Zero Switching Times

Consider the exhaustive regime with zero switching times. Say the server is faced at the initialization of the cycle at time t, with state

$$\underline{Q}(t) = (n_1, \ldots, n_k) \tag{26}$$

Consider the policy $\pi_0 = (1, 2, \ldots, K)$, whereby the server serves the channels via path π_0. The expected cycle time may be analytically solved, as in Browne and Yechiali [16] (1987-A). It is however interesting to derive π_0 directly via probabalistic reasoning.

As there are n_1 customers at channel 1, the expected server sojourn there is simply the sum of n_1 busy periods of type 1, $n_1 EV_1/(1-\rho_1)$, or a *delay* busy period of type 1 caused by service to the "original" n_1 customers. The expected service sojourn in channel 2 is the delay busy period of type 2 caused by the n_2 original customers plus the delay busy period of type 2 caused by the delay busy period of type 1, i.e.

$$\frac{n_2 EV_2}{1-\rho_2} + \frac{n_1 EV_1/(1-\rho_1)}{1-\rho_2} = \frac{n_2 EV_2}{1-\rho_2} + \frac{n_1 EV_1}{(1-\rho_1)(1-\rho_2)} \,. \tag{27}$$

Proceeding in this manner we may observe that the n_j original customers in channel j cause a delay busy period of type j in channel j with expected duration $\frac{n_j EV_j}{1-\rho_j}$, which directly causes a delay busy period of type $j+1$ to channel $j+1$ of expected duration $\frac{n_j EV_j/(1-\rho_j)}{1-\rho_{j+1}}$, which causes a delay busy period of type $j+2$ to channel $j+2$ with expected duration $\frac{n_j EV_j}{(1-\rho_j)(1-\rho_{j+1})}/(1-\rho_{j+2})$, etc. The *impact* of the original n_j customers in channel j on the entire expected cycle time following path π_0 is therefore seen to be

$$\frac{n_j EV_j}{(1-\rho_j)(1-\rho_{j+1})\cdots(1-\rho_K)} \,. \tag{28}$$

We immediately see that the expected cycle time following π_0, denoted by $C(\pi_0)$, can be decomposed into the *sum of the impacts caused by the initial customers present at t* i.e.,

$$\begin{aligned} C(\pi_0) = {} & \frac{n_1 EV_1}{(1-\rho_1)(1-\rho_2)\cdots(1-\rho_K)} + \frac{n_2 EV_2}{(1-\rho_2)(1-\rho_3)\cdots(1-\rho_K)} + \\ & + \cdots + \frac{n_j EV_j}{(1-\rho_j)(1-\rho_{j+1})\cdots(1-\rho_K)} + \cdots + \frac{n_K EV_K}{1-\rho_K} \,. \end{aligned} \tag{29}$$

Consider now the path $\pi_1 = (1, 2, \ldots, j-1, j+1, j, j+2, \ldots, K)$, i.e., π_1 consists of π_0 with the j^{th} and $j+1^{st}$ terms interchanged. The expected cycle time under π_1, denoted $C(\pi_1)$, now has the evaluation

$$\begin{aligned} C(\pi_1) = {} & \frac{n_1 EV_1}{(1-\rho_1)(1-\rho_2)\cdots(1-\rho_K)} + \cdots + \frac{n_{j+1} EV_{j+1}}{(1-\rho_{j+1})(1-\rho_j)(1-\rho_{j+2})\cdots(1-\rho_K)} \\ & + \frac{n_j EV_j}{(1-\rho_j)(1-\rho_{j+2})\cdots(1-\rho_K)} + \cdots + \frac{n_k EV_K}{1-\rho_K} \,. \end{aligned} \tag{30}$$

As we wish to determine the cycle path that minimizes cycle time, we calculate $C(\pi_0) - C(\pi_1)$. Upon manipulation, we find

$$C(\pi_0) - C(\pi_1) = \frac{n_j EV_j \rho_{j+1} - n_{j+1} EV_{j+1} \rho_j}{(1-\rho_j)(1-\rho_{j+1})\cdots(1-\rho_K)} \tag{31}$$

from which it is apparent that the switch is unprofitable (time-lengthening) i.e, $C(\pi_0) < C(\pi_1)$, if and only if

$$\frac{n_j EV_j}{\rho_j} < \frac{n_{j+1} EV_{j+1}}{\rho_{j+1}} \,,$$

equivalently, if and only if

$$\frac{n_j}{\lambda_j} < \frac{n_{j+1}}{\lambda_{j+1}} \,. \tag{32}$$

Upon repeated pairwise interchanges, we immediately deduce that

THEOREM 1:

For the *exhaustive* regime with zero switching times, the server faced with choosing the 'route' to *minimize* a cycle (time to serve all k-channels *once*) starting at time t from state $\underline{Q}(t) = (n_1, \ldots, n_k)$, should choose the route (path) based on *increasing* values of n_i/λ_i.

The optimal routing rule of Theorem 1 is *puzzling* in that it is seen that service times *play no role* in the determination of the optimal route!

3.2. Gated Regime, Zero Switching Times

Before we attempt any further analysis of the surprising result given by Theorem 1, we examine the *gated* regime with zero switching times. Recall that under gating, the server serves only those customers present in channel j upon his arrival to j during any cycle (tour). Consider once again a 'cycle' being determined at t based on $\underline{Q}(t)$, and following route π_0. We give here a direct probabilistic derivation, based on the analytic derivation of Browne and Yechiali [16] (1987-A). The server serves only the original n_1 customers in channel 1 so that his expected sojourn there is $n_1 EV_1$. During his sojourn at channel 1, an expected $\lambda_2(n_1 EV_1)$ customers arrived to channel 2, and the server must serve the original n_2 customers and those new arrivals to channel 2. The expected sojourn in channel 2 is therefore

$$n_2 EV_2 + \lambda_2(n_1 EV_1)EV_2 = n_2 EV_2 + \rho_2 n_1 EV_1 \ . \tag{33}$$

Similarly, the n_j original customers in channel j require expected service $n_j EV_j$, this lets in an expected $\lambda_{j+1}(n_j EV_j)$ customers to channel $j+1$, with expected service requirement $\rho_{j+1}(n_j EV_j)$. However, this delay (caused by the initial n_j customers in channel j) lets in an expected $\lambda_{j+2}(n_j EV_j + \rho_{j+1} n_j EV_j)$ customers into channel $j+2$, with expected service requirement $\rho_{j+2}(n_j EV_j + \rho_{j+1} n_j EV_j)$. The expected total delay caused by these n_j customers to channels $j, j+1$ and $j+2$ may therefore be written as

$$\begin{aligned} n_j EV_j + \rho_{j+1}(n_j EV_j) + \rho_{j+2}(n_j EV_j + \rho_{j+1}(n_j EV_j)) \\ = n_j EV_j (1+\rho_{j+1})(1+\rho_{j+2}) \ . \end{aligned} \tag{34}$$

Proceeding in this manner, we recognize that the total expected cycle time may be decomposed into the sum of the total delays caused by all the initial customers. Letting $T(\pi_0)$ denote the expected cycle time from $\underline{Q}(t)$ under the *gated* regime, we conclude that

$$\begin{aligned} T(\pi_0) = n_1 EV_1 \left[(1+\rho_2)(1+\rho_3)\cdots(1+\rho_K)\right] + n_2 EV_2 \left[(1+\rho_3)\cdots(1+\rho_K)\right] \\ + \cdots + n_j EV_j \left[(1+\rho_{j+1})\cdots(1+\rho_K)\right] + \cdots + n_K EV_K \ . \end{aligned} \tag{35}$$

Utilizing π_1 as before, we observe that

$$T(\pi_0) - T(\pi_1) = \left[n_j EV_j \rho_{j+1} - n_{j+1} EV_{j+1} \rho_j\right](1+\rho_{j+2})\cdots(1+\rho_K) \ .$$

Once again $T(\pi_0) < T(\pi_1)$ if and only if

$$\frac{n_j}{\lambda_j} < \frac{n_{j+1}}{\lambda_{j+1}} \ , \tag{36}$$

Leading to the *identical* form of the optimal server route:

THEOREM 2:

The server, in the *gated* regime system with zero switching times, minimizes the expected cycle time by following the route based on increasing values of n_i/λ_i.

Once again service times *play no role* in the determination of the optimal route!

3.3. Dynamic Routing

In practice the above results are directly implementable as they make use of only the on-line information $\underline{Q}(t)$, and $\underline{\lambda}$ (where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_K)$). One can envision a continuous routing policy (actually a dynamic priority rule) whereby the server may always choose the next channel to service, i.e., each channel completion affords the server a decision epoch. To be consistent with our previous definition of pseudo-cyclic policies, the server dichotomizes the set of channel indices into two groups; $A = \{$channels *served* on this tour$\}$, $\overline{A} = \{$channels *unserved* on this tour$\}$, where a "tour" consists of a route visiting each channel at most once. At each channel completion, the index of that channel is placed into A and the server chooses the next channel from $\overline{A}$. When $\overline{A}$ is empty, A is *now* labelled $\overline{A}$ and vice versa. The optimal routing policy under this scenario can be seen (see Browne and Yechiali [16] (1987-A) to be equivalent to the server always choosing to service that channel with *minimal* value of $n_i(t)/\lambda_i; i \in \overline{A}, t$ a decision epoch, where $n_i(t)$ is the *current* number of occupants in channel i.

3.4. Swap-in and Switch-out Times

When switching times between channels are *non-zero*, the analysis is only slightly more complicated, but it turns out that the service times as well as the switching times *do play a role* in the determination of the optimal routes.

Letting S_i be the random time it takes the server to *switch-out* of channel i, and P_j be the random time it takes the server to switch-into, or *swap-in* to channel j, we may simplify the form of a general switching time from channel i to channel j, S_{ij}, by assuming they combine *additively*, i.e., we assume $S_{ij} = S_i + P_j$, where S_i, P_j are independent, $\forall i, j$. Under this assumption, Browne and Yechiali [16] (1987-A) have shown that the above discussion holds true when the server chooses the *next* channel with minimal

$$\frac{n_i(t)EV_i + EP_i + (1 - \rho_i)ES_i}{\rho_i}, \tag{37}$$

for the *exhaustive* regime, and with minimal

$$\frac{n_i(t)EV_i + (1 + \rho_i)EP_i + ES_i}{\rho_i} \tag{38}$$

for the *gated* regime. Decision epochs now occur when the server has completed *switching-out* of a channel.

It may further be seen from Browne and Yechiali [17] (1987-B) that these routes correspond to the Gittins Index (see Whittle [18] (1982)) for the problem of minimizing each tour (or 'pseudo-cycle') time.

4. CONCLUSION

Rules (37) and (38) minimize "tour" times, or "Pseudo-cyclic" times for the exhaustive and gated regimes in a polling system with swap-in and switch-out times. These rules, or server routes, are dynamic, adaptive and of a simple form amenable to direct implementation. A study of the impact of these dynamic routes on waiting times of customers and the connection with the optimality equations (24) and (25) is the next step in the analysis of control policies for polling systems.

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